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ASYMPTOTIC STUDY OF TOEPLITZ DETERMINANTS WITH FISHER-HARTWIG SYMBOLS AND THEIR DOUBLE-SCALING LIMITS

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Abstract

This thesis aims to study the asymptotic behavior of Toeplitz determinants $D_n(f_t(z))$ by using the Riemann-Hilbert analysis. We consider the double scaling limits of Toeplitz determinants with respect to symbol $f_t(z)$. This symbol possess m Fisher-Hartwig singularities when $t > 0$, and $m+1$ if $t \to 0$. We obtain the uniform asymptotics for $D_n(f_t(z))$ as $n \to \infty$ which is valid for all sufficiently small t in terms of Painlevé V function. This study is divided into two parts: We first consider the case when the seminorm $|||\beta^{(t)}||| < 1$ for $t \geq 0$ and then the case of the Basor-Tracy asymptotics when $|||\beta^{(t)}||| = 1$ for some t. The latter case is further divided to the cases, $|||\beta^{(t)}||| < 1$ for $t > 0$ and $|||\beta^{(t)}||| = 1$ for $t > 0$.

In the last chapter we present the computation of the magnetization of the 2D Ising model in the high temperature regime $T > T_c$ (i.e., $t < 0$) including all the details by using the Riemann-Hilbert approach and the asymptotics of Toeplitz determinants.

Declaration

I confirm that this is my original work and that all outside material has been properly cited and acknowledged.

Reham Alahmadi

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List of Symbols

Roman Symbol	Description	References
$a(z,t)$		Equation 2.7.1
$A_k, A_{e^{\pm t}}$		Equation (3.4.159 - 3.4.161)
$A(z)$		Equation 3.5.6
$b_{\pm}$	Wiener-Hopf factorisation	Equation 2.4.13
$C(\alpha,\beta)$		Equation 2.7.5
$(Cf)(z)$	The cauchy integral type	Equation 2.6.1
$D_n(f)$	The Toeplitz determinants	Equation 2.2.4
$D(z)$	The Szegő function	Equation 3.4.11
$E[f]$		Equation 2.3.2
$E(\sigma)$	Total interaction energy	Equation 4.2.1
$f(z; n_0, n_1, ..., n_m)$	Fisher and Hartwig representation	
f_n	The Fourier coefficients of the function f	Equation 2.1.1
$f(z,t)$	Toeplitz matrix symbol	Equation 3.2.1
$\|\|f\|\|_p$	The norm in the p measurable space	Definition 2.1.1
F_j		Equation 3.4.42
F_n		Equation 3.6.5
g_{z_j,β_j}	The jump function	Equation 2.4.2
$g_t(z)$		Equation 3.5.31
$G(z)$		Equation 3.4.91
$G[f]$		Equation 2.3.2
$G_{\alpha_j+\beta_j,\alpha_j-\beta_j}$		Equation 2.4.12
H^p	The hardy space	Definition 2.1.2
h_{α_j}		Equation 3.4.19
$K(x)$		Equation 3.4.129

Roman Symbol	Description	References
L^p	The p measurable space,	Definition 2.1.1
l^p	The spaces of all sequences	Definition 2.1.1
L^∞	The measurable space of essentially bounded functions	Definition 2.1.1
M	The spontaneous magnetization	Equation 4.2.7
M_ϕ	The multiplication operator	Definition 2.2.2
$M(\tilde{\lambda})$	The solution of Riemann Hilbert problem for M	
$N(z)$	The solution of the Riemann Hilbert problem for $N(z)$	
O_β	Orbit of β	Equation 2.4.17
$\mathcal{O}, o$	Big and Small 'o' notation	Definition 2.3.1
P	The Orthogonal projection	Equation 2.2.1
P_{z_j}	Local paramtreices at z_j	Equation 3.4.40
P_{z_0}	Local paramtreices at $z_0 = 1$	Equation 3.4.97
$R(z)$	Small norm Riemann Hilbert problem	
$\tilde{R}(\tilde{\lambda})$	The solution of the Riemann Hilbert problem for $\tilde{R}(\tilde{\lambda})$.	
$\hat{R}(z)$	The solution of the Riemann Hilbert problem for $\hat{R}(z)$	
$R_n(t)$		Equation 3.5.29
$S(z)$	The solution of the Riemann Hilbert problem for $S(z)$	
T_f	Toeplitz operator	Equation 2.2.2
$T(f)$	The infinite Toeplitz matrix	Definition 2.2.4
$T_n(f)$	The finite Toeplitz matrix	Definition 2.2.5
$\mathbb{T}$	The unit circle	
$T(z)$	The solution of the Riemann Hilbert problem for $T(z)$.	
U_{z_j}	small neighborhood around z_j	Equation 3.4.10
$V(z)$		Equation 2.4.13
$v(x)$		Equation 3.5.22
v_j		Equation 3.6.32
$V_T(z)$	the jump function of RH-T2	Equation 3.4.7
$W(z)$		Equation 3.4.98
$w(x)$		Equation 3.5.18
$X(z)$	The solution for X Riemann Hilbert problem	
$Y(z)$	The solution of the Riemann Hilbert problem for $Y(z)$	Equation 3.3.1
$Z(T)$	The partition function	Equation 4.2.2

Greek Symbol	Description	References
$\delta_j(z)$	The asymptotic sequences	Definition 2.3.2
β_j	jump type singularities	Equation 2.4.1
$\beta(z)$	Szegő function with respect to Szegő smbol	Equation 4.2.27
α_j	Root type singularity	Equation 2.4.1
ϕ_{z_j,β_j}		Equation 2.4.4
$\|\|\|\beta\|\|\|$	β seminorm	Equation 2.4.10
$\eta(z;t)$		Equation 4.1.1
$\hat{\eta}(z;t)$		Equation 4.2.14
$\phi_n(z), \hat{\phi}_n(z^{-1})$	The orthogonal polynomials	Equation 2.5.1
$\phi_{ons}(e^{i\theta}$	Onsager function	Equation 4.2.9
$\Gamma(z)$	The Euler's $\Gamma-$ function	
χ_n	The leading coefficients	Equation 2.5.6
$\Omega(2nt)$		Equation 2.7.3
$\tilde{\Omega}(2nt)$		Equation 3.1.3
$\sigma(x)$	Painlevé V equation	Equation 2.7.4
$\Gamma(z)$	The Euler's $\Gamma-$ function	
$\Sigma_1(t)$		Equation 3.1.5
ξ		Equation 3.4.41
Ψ_j	The solution of Riemann Hilbert problem for Ψ_j	
Ψ	The solution of Riemann Hilbert problem for Ψ	
$\tilde{\Psi}$	The solution of Riemann Hilbert problem for $\tilde{\Psi}$	
$\Phi(z)$		Equation 3.4.90
$\hat{\Phi}(\xi)$		Equation 3.4.113
$\lambda(z)$		Equation 3.4.95
σ_1, σ_3	Pauli matrices	
η_j		Equation 3.4.66
Δ_1	The first correction term	Equation 3.4.73
ω		Equation 3.4.148

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Chapter 1

Introduction

The aim of this thesis is to examine the asymptotic behaviour of Toeplitz determinants $D_n(f_t)$ by using the Riemann-Hilbert approach, which has been used for solving many asymptotic problems. The spontaneous magnetization problem for the 2D Ising model is a popular example of the application of these types of problems in the field of statistical mechanics.

The second chapter presents a comprehensive examination of the related mathematical principles derived from functional analysis, alongside a historical context and established methodologies employed in addressing the problem discussed in subsequent chapters.

In Chapter 3, double-scaling limits of Toeplitz determinants, defined by the Painlevé V function $\sigma(x)$, are studied. Claeys, Its, and Krasovsky in [13] discussed the role of the function $\sigma(x)$ in the asymptotic expansions. When $t > 0$, the symbol f_t possesses m Fisher-Hartwig singularities with parameters $\alpha_j \in \mathbb{C}$ and $\beta_j \in \mathbb{C}$, $j = 1, ..., m$, and when $t \to 0$, the symbol possesses $m + 1$ Fisher-Hartwig singularities as a new singularity emerges at $z = 1$. For sufficiently small t, we obtain uniform asymptotic of $D_n(f_t)$ as $n \to \infty$. The focus of our work involves the relation between orthogonal polynomials and Riemann-Hilbert problems that are linked to our symbol. The Riemann-Hilbert problem and Toeplitz determinants will then be related through the use of a differential identity. We look into two possibilities because our symbol has at least m jump singularities when $t \to 0$. Applying concepts from [18] and [34], we analyze the seminorm $|||\beta|^{(t)}|| < 1$ in the first part of this chapter. The expression we have here defines the Toeplitz determinant with m Fisher-Hartwig singularities for $t > 0$ and the determinant with $m + 1$ singularities for $t \to 0$. Ehrhardt provides the identical solution for both of these asymptotic regimes, but with different parameters and, hence, different asymptotics, as seen in [22]. In addition, we consider the seminorm where $|||\beta^{(t)}||| = 1$, in which case, we have different possible subcases. In the first case we have $||||\beta^{(t)}||| < 1$ if $t > 0$, and when $t \to 0$, we have $|||\beta^{(t)}||| = 1$. This case describes the transition between the asymptotic regime for Toeplitz determinants whose symbol has m singularities and does not contain Fisher-Hartwig representations and the asymptotic regime for Toeplitz determinants with Fisher-Hartwig representations. Thus by assuming this, the asymptotic behaviour of Toeplitz determinants can be expressed as a linear combination of the results obtained by Ehrhardt. We have two types of Fisher-Hartwig representations, the non-trivial Fisher-Hartwig representation if $\Re\beta_0 < \Re\beta_1 \leq \ldots \leq \Re\beta_{m-1} < \Re\beta_m$, and the trivial Fisher-Hartwig representation if there is an $l \geq 1$, such that $\Re\beta_0 < \Re\beta_1 \leq ... \leq \Re\beta_{m-l} < \Re\beta_{m-l+1} = ... = \Re\beta_m$. In the second subcase, we have the transition from the asymptotic regime of a determinant with a symbol containing Fisher-Hartwig representations with m singularities to the asymptotic regime of a determinant with Fisher-Hartwig representations with $m + 1$ singularities in the symbol. In the last section we discuss the remaining cases related to the double-scaling limits involving the Basor-Tracy conjecture and also matrix-valued symbols.

In Chapter 4, we compute the magnetization of the 2D Ising model in the high temperature regime $T > T_c$ using the Riemann-Hilbert approach and the asymptotics of Toeplitz determinants $D_n(f_t)$ with one generate Fisher-Hartwig singularity at $z = 1$, $\alpha = 0$ and $\beta = -1$ in more details. Previously, in [49] and [41], this result was obtained using physical arguments and in [17] , the authors describe the analysis in words and claim the result. However, here we provide more details of the first order term using the Riemann-Hilbert analysis. The last section of this chapter contains open problems related to the transition from the high temperature regime $T > T_c$ to the critical temperate $T = T_c$.

Chapter 2

Preliminaries

In this chapter, we present the necessary mathematical foundations that will be used in the rest of the thesis. We will focus primarily on the analyticity of functions. Simply, the function $f(z)$ is called analytic at z if it is differentiable in a neighborhood of z, and called analytic in the region U if it is analytic at every point $z \in U$. The analytic function in the neighborhood of z can be expressed as a Taylor series at every point $z \in U$.

2.1 Hardy spaces and L^p spaces

Definition 2.1.1. *Let $1 \leq p < \infty$. We denote by $L^p(\mathbb{T})$ the space of all complex-valued measurable functions on the unit circle $\mathbb{T} = \{|w| = 1\}$ with the norm:*

$$||f||_p = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(e^{i\theta})|^p d\theta \right\}^{1/p}.$$

The following are examples of particular L^p spaces:

1. $L^\infty(\mathbb{T})$ is the space of all measurable and essentially bounded complex-valued functions, with the norm:

$$||f||_\infty = \inf\{C > 0 : \quad |f| \leq C \quad \text{almost everywhere}\} < \infty.$$

2. $L^2(\mathbb{T})$ space is a Hilbert space of sequre-integrable function $f : \mathbb{T} \to \mathbb{C}$ with inner

product $(f,g) = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}\overline{g(e^{i\theta|})}\, d\theta$, and the norm given by

$$||f||_2^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\, d\theta.$$

3. For $0 < p < \infty$, l^p is the space of all sequences $x = (x_i)_{i\in\mathbb{N}}$ with the norm:

$$||x|| = \left(\sum_{i\in\mathbb{N}} |x_i|^p\right)^{1/p} < \infty.$$

We define the Fourier coefficients f_n of a function $f \in L^1(\mathbb{T})$ by

$$f_n = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-in\theta} d\theta. \tag{2.1.1}$$

Definition 2.1.2. *For $1 \leq p \leq \infty$, the Hardy space H^p is defined by*

$$H^p(\mathbb{T}) = \{f \in L^p(\mathbb{T}) : f_n = 0 \text{ for all } n < 0\}.$$

For more information on the Hardy spaces and L^p, we refer to [3], [36], and [39].

2.2 Toeplitz Determinants

A vast array of mathematical and physical problems can be formulated using Toeplitz matrices and Toeplitz determinants. For specifics on the theory and applications of Toeplitz determinants, see [11], and the more recent survey paper [17]. We start by providing brief definitions of Toeplitz operators and multiplication operators.

Observe first that $H^2(\mathbb{T})$ is a closed subspace of $L^2(\mathbb{T})$. The orthogonal projection P of $L^2(\mathbb{T})$ onto the space $H^2(\mathbb{T})$ can be expressed as follows:

$$P\Big(\sum_{n=-\infty}^{\infty} f_n e^{in\theta}\Big) \to \sum_{n=0}^{\infty} f_n e^{in\theta}. \tag{2.2.1}$$

The orthogonal projection P is referred to as the Riesz projection.

Definition 2.2.1. *For $1 \leq p < \infty$ and $\phi \in L^\infty(\mathbb{T})$, the multiplication operator $M_\phi : L^p(\mathbb{T}) \to$*

$L^p(\mathbb{T})$ *is defined as*

$$M_\phi f = \phi f.$$

Definition 2.2.2. *For $f \in L^\infty(\mathbb{T})$ and $1 < p < \infty$, the Toeplitz operator T_f is defined on $H^p(\mathbb{T})$ by*

$$T_f = PM_f. \tag{2.2.2}$$

Let $1 < p < \infty$. Since P is bounded from $L^p(\mathbb{T})$ to $H^p(\mathbb{T})$, it follows that T_f is bounded on $H^p(\mathbb{T})$.

Definition 2.2.3. *The Toeplitz matrix corresponding to the symbol $f \in L^1(\mathbb{T})$ with respect to the basis $\{e^{in\theta}\}$ is given by:*

$$T(f) = (f_{j-k})_{j,k\geq 0} = \begin{pmatrix} f_0 & f_{-1} & f_{-2} & f_{-3}\cdots \\ f_1 & f_0 & f_{-1} & f_{-2}\cdots \\ f_2 & f_1 & f_0 & f_{-1}\cdots \\ \vdots & \vdots & \ddots & \vdots \end{pmatrix}, \tag{2.2.3}$$

where f_n is the nth Fourier coefficient of f.

Remark 2.2.4. *A bounded operator on l^2 is generated by the Toeplitz matrix (2.2.3) if and only if $f \in L^\infty(\mathbb{T})$. The Toeplitz operator T_f on the sequence space $l^2(\mathbb{Z}_+)$ is represented by these matrices with respect to the standard basis $\{e_n = e^{in\theta} : n \geq 0\}$. One way to observe this is to let $f \in L^\infty(\mathbb{T})$, and then notice that*

$$T_f e_k = P \sum_{n=-\infty}^{\infty} f_n e^{in\theta} e^{ik\theta} = \sum_{r=0}^{\infty} f_{r-k} e^{ir\theta} \quad , r = n + k$$

or by using the inner product in L^2 as follows:

$$\begin{aligned}(T_f e_k, e_n) &= (PM_f e_k, e_n) = (fe_k, P^* e_n) = (fe_k, e_n) \\ &= \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) e^{ik\theta} e^{-in\theta} d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta}) e^{(k-n)i\theta} d\theta \\ &= f_{n-k}\end{aligned}$$

Definition 2.2.5. *The $n \times n$ finite Toeplitz matrix $T_n(f)$ is given by,*

$$T_n(f) = (f_{j-k})_{0\le j,k\le n-1} = \begin{pmatrix} f_0 & f_{-1} & f_{-2} & \cdots & f_{-(n-1)} \\ f_1 & f_0 & f_{-1} & \cdots & f_{-(n-2)} \\ \vdots & \vdots & \ddots & \vdots & \\ f_{n-1} & f_{n-2} & f_{n-3} & \cdots & f_0 \end{pmatrix}. \tag{2.2.4}$$

Moreover, $D_n(f) = \det T_n(f)$ denotes the Toeplitz determinant.

Infinite Toeplitz matrices $T(f)$ play an important role in operator theory and functional analysis, whereas truncated Toeplitz matrices $T_n(f)$ have made considerable contributions to several areas of mathematics physics, engineering, and linear algebra. Our goal is to study what happens to Toeplitz matrices as their size tends to infinity.

2.3 Asymptotic behavior of Toeplitz determinants

Since we aim to study the asymptotic behavior of Toeplitz determinants $D_n(f)$ as the size of the matrix increases to infinity. It is important to introduce the following concepts that will be utilized throughout.

Definition 2.3.1. *1. The notation $f(z) = \mathcal{O}(g(z))$ as $z \to z_0$ (i.e., $f(z)$ is of order $g(z)$) means there exists a finite constant $K > 0$ such that $|f(z)| \le K|g(z)|$ for all z in a neighborhood of z_0.*

2. Asymptotically smaller: the notation

$$f(z) = o(g(z)) \quad \text{as} \quad z \to z_0,$$

means that

$$\lim_{z\to z_0} \left|\frac{f(z)}{g(z)}\right| = 0$$

3. Asymptotically equal: The notation $I(z) \sim g(z)$ as $z \to z_0$, means

$$\lim_{z\to z_0} \left|\frac{I(z)}{g(z)}\right| = 1.$$

In what follows, we will consider two types of symbols, beginning with Szegő, a symbol $f(z)$ that satisfies a certain smoothness condition, has no winding around the origin, no-zeroes on the unit circle $\mathbb{T}$, and has an analytic continuation in the annulus of $\mathbb{T}$. This will be followed by the introduction of the Fisher-Hartwig symbols which, instead of being continuous or analytic functions, could have zeros, integrable singularities, and a non-zero winding number.

2.3.1 Szegő symbols

The story of studying the asymptotics of Toepitz determinants with the Szegő symbols starts in 1915 when Szegő proved the following theorem, which George Polya had conjectured.

Theorem 2.3.2 (First Szegő Limit theorem [47])**.** *Let $f > 0$ be a positive function, continuous on the unit circle $\mathbb{T}$. Then*

$$\frac{1}{n} \log D_n(f) = (\log f)_0, \tag{2.3.1}$$

which can written as

$$D_n(f) = \exp\{n(\log f)_0 + o(n)\}. \tag{2.3.2}$$

In the 1940s, Onsager and Bruria Kaufman inspired him to revisit his calculations from nearly four decades prior. The following is a well-known result that is still useful today and has been generalized many times by a large number of mathematicians. Numerous mathematical proofs from various fields can be found in Barry Simon's OPUC book [46], (also see [10] and [11]).

Theorem 2.3.3. *(Szegő Strong Limit Theorem (SSLT) [25]). If $\ln f(z)$ is sufficiently smooth on the unit circle $\mathbb{T}$ (specifically, $f(z)$ is the szegő symbol), then $\ln f(z) \in L^1(\mathbb{T})$ and the subsequent sum converges.*

$$\sum_{k=-\infty}^{\infty} |k||(\ln f)_k|^2, \quad (\ln f)_k = \frac{1}{2\pi} \int_{\mathbb{T}} \ln f(e^{i\theta}) e^{-ik\theta} d\theta. \tag{2.3.3}$$

Then the following expression for the Toeplitz determinants:

$$\ln D_n(f) = \frac{n}{2\pi}\int_{\mathbb{T}} \ln f(e^{i\theta})d\theta + \sum_{k=1}^{\infty} k(\ln f)_k(\ln f)_{-k} + o(1). \tag{2.3.4}$$

2.4 Fisher-Hartwig symbols and their conjecture

In the 1960s, a variety of problems in statistical mechanics led to the consideration of symbols for Toeplitz matrices with a higher level of complexity. These new symbols had zeros, integrable singularities, and non-zero winding numbers in place of continuous or analytic functions. Michael Fisher and Robert Hartwig had found an efficient technique for factorizing these singularities; see [23]. These symbols are called Fisher-Hartwig symbols and defined as follows:

$$f(z) = e^{V(z)} z^{\sum_{j=0}^{m}\beta_j} \prod_{j=0}^{m} |z-z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \quad \theta \in [0,2\pi) \tag{2.4.1}$$

for some $m \geq 0$, $z_j = e^{i\theta_j}, \quad \theta_j \in (0,2\pi)$,$\beta_j \in \mathbb{C}$, $\Re\alpha_j > -\frac{1}{2}$, and

$$g_{z_j,\beta_j(z)} = \begin{cases} e^{i\pi\beta_j} & , \text{ if } \theta \in [0,\theta_j) \\ e^{-i\pi\beta_j} & , \text{ if } \theta \in [\theta_j, 2\pi) \end{cases} \tag{2.4.2}$$

where $V(z)$ is analytic in a neighborhood of the unit circle $\mathbb{T}$. The symbol has $m+1$ singularities at $z_j = e^{i\theta_j}, j = 0,...,m, \theta_j \in [0,2\pi)$ and the condition $\Re\alpha_j > -1/2$ for integrability. If $\alpha_j \in \mathbb{C}$ is not zero, the root type exists, and if $\beta_j \neq 0$ for $\beta_j \in \mathbb{C}$ the jump type singularities are defined. In this thesis, it will be assumed that $V(z)$ is analytic in the neighborhood of the unit circle. However, a result has been demonstrated in [16] for the more general $V(z)$, where the function is less smooth.

Remark 2.4.1. *The Fisher-Hartwig symbols can also be written in other ways. The Riemann-Hilbert analysis use the symbols in the form (2.4.1), while the operator-theoretic methods use the following form:*

$$f(z) = b(z)\prod_{j=1}^{m} |z-z_j|^{2\alpha j}\phi_{z_j,\beta_j}(z) \quad (z \in \mathbb{T}), \tag{2.4.3}$$

where $\beta \in \mathbb{C}$, $z_j \in \mathbb{T}$, *and* ϕ_{z_j,β_j} *are defined as*

$$\phi_{z_j,\beta_j} = \exp\left\{ i\beta_j \arg(-\frac{z}{z_j})\right\} \quad \textit{with} \quad \arg z \in (-\pi, \pi). \tag{2.4.4}$$

It is easy to show the equivalence between the two definitions. The authors using Riemann-Hilbert approach have modified $\arg z$, which has a significant impact on the function $f(z)$, and eliminated some of its components. The $z_j^{-\beta_j}$ factors have been removed so that the work can be more easily compared to others in the literature, and the $z^{\sum_{j=0}^{\infty}\beta_j}$ factors are removed so that the work can be more accurately characterized by the Tracy-Basor conjecture in [16]. For example, in (2.4.3), the function $b(z)$ and the function $e^{V(z)}$ in (2.4.1) are similar.

2.4.1 Fisher-Hartwig asymptotics

The asymptotic analysis of Toeplitz determinants with Fisher-Hartwig symbols has been improved by multiple studies. To learn more about history see, [17]. In [37] and [38], Lenard provided the full asymptotics for symbols exhibiting zeros on the unit circle with $\beta_j = 0$ and $\alpha_j > -1/2$. Fisher-Hartwig in [23] proposed a conjecture on the asymptotic behavior of Toeplitz determinants, specifically in relation to symbols described in equation (2.4.3). This conjecture was inspired by the research done by Wu [49] and potentially influenced by Lenard's calculations. In [48], Widom confirmed Lenard's conjecture for $\alpha_j \in \mathbb{C}$, with $\Re\alpha_j > -1/2, \beta_j = 0$ and $V(z)$ is smooth, also he found an explicit description for the constant $E(z)$. Estelle Basor and J. William Helton evaluated pure Fisher-Hartwig singularities using repeated approaches [6]. Böttcher and Silbermann derived in [9], a specific equation for the determinant that contains these pure Fisher-Hartwig symbols. In his 1997 doctoral dissertation, Ehrhardt [22] computed explicitly the asymptotics of the leading term of the Toeplitz determinants with Fisher-Hartwig symbols in (2.4.1).

The pure Fisher-Hartwig symbol

This function does not contain a nice function $e^{V(z)}$, and has the following form

$$w_{\alpha,z_j}(z)\phi_{\beta,z_j}(z) = (1 - \frac{z_j}{z})^{\alpha-\beta}(1 - \frac{z}{z_j})^{\alpha+\beta}, \tag{2.4.5}$$

where ϕ_{β,z_j} is defined in (2.4.4), and $w_{\alpha,z_j} = |e^{i\theta} - e^{i\theta_j}|^{2\alpha} = 2^{2\alpha}|\sin\frac{\theta-\theta_j}{2}|^{2\alpha}$.

$$D_n(f) = \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}\frac{G(n)G(n+2\alpha)}{G(n+\alpha+\beta)G(n+\alpha-\beta).} \tag{2.4.6}$$

If $\alpha \pm \beta \in \mathbb{Z}_-$, then $D_n(f)$ equals zero. The Barnes G-function is defined by

$$G(z+1) = (2\pi)^{z/2}e^{-z(z+1)/2-Cz^2/2}\prod_{n=1}^{\infty}\left\{\left(1+\frac{z}{n}\right)^n e^{-z+z^2/(2n)},\right\} \tag{2.4.7}$$

where $C = 0.577...$ is the Euler's constant. $G(z+1) = \Gamma(z)G(z)$ is an identity that illustrates the similarity between the Barnes $G(z)$ and $\Gamma(z)$ functions. From (2.4.7), we can also derive the following asymptotic behavior as $n \to \infty$:

$$\frac{G(n)G(n+\gamma+\delta)}{G(n+\gamma)G(n+\delta)} \sim n^{\gamma\delta}, \quad \delta, \gamma \in \mathbb{C}. \tag{2.4.8}$$

Thus the asymptotics of Toeplitz determinants with pure Fisher-Hartwig singularities in (2.4.6) can be written as follows,

$$D_n(f) \sim \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}n^{\alpha^2-\beta^2}. \tag{2.4.9}$$

The result on Fisher-Hartwig asymptotics was proved first by Ehrhardt when $|||\beta||| < 1$ using operator theory (see, [22]), and subsequently, it was proved in full generality in [16] using the Riemann-Hilbert method. Before stating Ehrhardt's result, the seminorm is defined by:

$$|||\beta||| = \max_{1\le j,k\le m} |\Re\beta_j - \Re\beta_k|, \tag{2.4.10}$$

where $\alpha_0 = \beta_0 = 0$, and $|||\beta||| = 0$ if we have only one singularity, i.e., $m = 0$.

Theorem 2.4.2. *Let f be defined as in (2.4.1) with $V \in C^\infty$, $|||\beta||| < 1$ and $\alpha_j \pm \beta_j \notin \mathbb{Z}_-$*

for $j = 0, ..., m$. Then as $n \to \infty$

$$D_n(f) = \exp\left\{nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k}\right\} \prod_{j=0}^{m} b_+(z_j)^{-(\alpha_j-\beta_j)} b_-(z_j)^{-(\alpha_j+\beta_j)}$$
$$\times n^{\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)} \prod_{0\le j<k\le m} |z_j - z_k|^{2(\beta_j\beta_k-\alpha_j\alpha_k)} \left(\frac{z_k}{z_j e^{i\pi}}\right)^{(\alpha_j\beta_k-\alpha_k\beta_j)} \quad (2.4.11)$$
$$\times \prod_{j=0}^{m} G_{\alpha_j+\beta_j,\alpha_j-\beta_j}(1+o(1)),$$

and

$$G_{\alpha_j+\beta_j,\alpha_j-\beta_j} = \frac{G(1+\alpha_j+\beta_j)G(1+\alpha_j-\beta_j)}{G(1+2\alpha_j)}. \quad (2.4.12)$$

Here the Fourier expansion of the function $V(z)$ on the unit circle is denoted by

$$V(z) = \sum_{k=-\infty}^{\infty} V_k z^k, \quad \text{where} \quad V_k = \frac{1}{2\pi}\int_0^{2\pi} V(e^{i\theta})e^{-ik\theta}d\theta, \quad (2.4.13)$$

and the Wiener-Hopf factorisation of the function $e^{V(z)}$ is defined by

$$e^V(z) = b_+(z)e^{V_0}b_-(z), \quad b_+ = \exp\sum_{k=1}^{\infty} V_k z^k, \quad b_- = \exp\sum_{k=-\infty}^{-1} V_k z^k. \quad (2.4.14)$$

In [16] and [18], the authors re-prove Theorem 2.4.2 using the Riemann-Hilbert problem. In addition, they make it work for functions $V(z)$ with less smoothness that satisfies the following conditions:

$$\sum_{k=-\infty}^{\infty} |k|^s |V_k| < \infty,$$

for some s such that

$$s > \frac{1 + \sum_{k=0}^{m}[(\Im\alpha_k)^2 + (\Re\beta_k)^2]}{1 - |||\beta|||}.$$

2.4.2 Basor-Tracy conjecture.

The symbol $f(e^{i\theta})$ with two jump singularities, at $z_0 = 1$, and $z_1 = -1$, were examined by Basor and Tracy in [7], who considered the symbol

$$f(e^{i\theta}) = g_{1,1/2}(z)g_{-1,-1/2}(z)e^{i\pi/2},$$

with $\beta_0 = 1/2$ and $\beta_1 = -1/2$, which means we have $|||\beta||| = 1$. By performing direct calculations, they determined that as n approaches infinity,

$$D_n(f) = \frac{1+(-1)^n}{2}\sqrt{2/n}G(1/2)^2G(3/2)^2(1+\mathcal{O}(1)).$$

The asymptotics for $D_nf(z)$ were not of the general standard Fisher-Hartwig form. They noticed that there is another Fisher-Hartwig representation for $f(e^{i\theta})$, where $\beta_0 = -1/2$ and $\beta_1 = 1/2$. Thus, they determined that the asymptotics they obtained were actually given by a combination of two Fisher-Hartwig asymptotic forms (2.4.11), one asymptotic form corresponded to the symbol as $|||\beta||| < 1$, while the other was the Fisher-Hartwig singularity for the symbol with $z_0 = 1$ and $z_1 = -1$ but with $\beta_0 = -1/2$ and $\beta_1 = 1/2$. The only difference between the second and original symbol was a constant. This holds true for every Fisher-Hartwig symbol.

2.4.3 Fisher-Hartwig representation.

Assume that we have a Fisher-Hartwig symbol $f(z)$ in (2.4.1) with $\beta_j \neq 0$ or $\alpha_j \neq 0$ or both. Then by replacing β_j with $\beta_j + n_j = \hat{\beta}_j$, where $\sum_{j=0}^{m} n_j = 0$ and let $\{n_0, n_1, ..., n_m\} \in \mathbb{Z}$, we obtain the function $f(z; n_0, ..., n_m)$ which is called a Fisher-Hartwig representation of $f(z)$. Noting that the multiplicative constants are the only distinction between all Fisher-Hartwig representations of $f(z)$,

$$f(z) = \prod_{j=0}^{m} z_j^{n_j} \times f(z; n_0, ..., n_m). \qquad (2.4.15)$$

We are interested in Fisher-Hartwig representations where

$$\sum_{j=0}^{m}(\Re\beta_j + n_j)^2, \tag{2.4.16}$$

is minimal. There exists a finite number of such representations. There is a particular way to find them, which was described in [16]. The group of all forms in (2.4.16) is denoted by M. The representation of the Fisher-Hartwig is said to be degenerate if $\alpha_j \pm (\beta_j + n_j) \in \mathbb{Z}_-$ for some j. Let us use the following to describe the Orbit of $\beta = (\beta_0, ..., \beta_m)$

$$O_\beta = \Big\{\hat{\beta} : \hat{\beta}_j = \beta_j + n_j, \sum_{j=0}^{m} n_j = 0\Big\}. \tag{2.4.17}$$

The following is a way to describe the set M.

Lemma 2.4.3 (Lemma1.12 [16]). *Only the following two alternatives are possible:*

1. *There is $\hat{\beta} \in O_\beta$ such that $|||\hat{\beta}||| < 1$. Then $\hat{\beta}$ is unique, and it is the unique element of $M = \{\hat{\beta}\}$.*

2. *There is $\hat{\beta} \in O_\beta$ such that $|||\hat{\beta}||| = 1$. Then there are at least two of these $\hat{\beta}$'s, and they all come from each other by applying the following rule repeated by adding 1 to a $\hat{\beta}_j$ with the smallest real part and subtract 1 from a $\hat{\beta}_j$ with the biggest real part. In addition, $M = \{\hat{\beta} \in O_\beta : |||\hat{\beta}||| = 1\}$.*

Deift, Its, and Krasovsky in [16] proved the Basor-Tracy conjecture, which is the following theorem,

Theorem 2.4.4. *(Basor-Tracy conjecture) Consider the function $f(z)$ defined by (2.4.1), where $\Re\alpha_j > \frac{-1}{2}, \beta_j \in \mathbb{C}, j = 0, ..., m$, and suppose M is not degenerate. Then*

$$D_n(f(z)) = \sum \left(\prod_{j=0}^{m} z_j^{n_j}\right)^n R(f(z; n_0, ..., n_m))(1 + o(1)), \quad \textit{as} \quad n \to \infty, \tag{2.4.18}$$

where the sum represents all Fisher-Hartwig representations in M. Excluding the error term , every $R(f(z; n_0, ..., n_m))$ represents the right side of the formula (2.4.11).

2.5 The relation between Toeplitz determinants and orthogonal polynomials

Let $f(z)$ be a complex-valued function on the unit circle $\mathbb{T}$. In this section, we will describe the connection between Toeplitz determinants $D_n(f)$ and the system of orthogonal polynomials (OPs) with weight $f(z)$. The orthogonal polynomials $\phi_n(z) = \chi_n z^n + ...$, and $\hat{\phi}_n(z) = \chi_n z^n + ...$ of degree n provide an important role in the asymptotic study of the Toeplitz determinants. Assume that $D_n(f) \neq 0$, for $n = n_0, n_0+1, ...$ for some sufficiently large n_0. Then by this condition, the polynomials $\phi_n(z) = \chi_n z^n + ...$, and $\hat{\phi}_n(z) = \chi_n z^n + ...$ of degree n exist and satisfy the orthogonality conditions

$$\int_{\mathbb{T}} \phi_n(z) z^{-j} f(z) \frac{dz}{2\pi iz} = \chi_n^{-1}\delta_{jn} \quad \int_{\mathbb{T}} \hat{\phi}_n(z^{-1}) z^{j} f(z) \frac{dz}{2\pi iz} = \chi_n^{-1}\delta_{jn}, \tag{2.5.1}$$

which alternatively can be written as

$$\int_{\mathbb{T}} \phi_n(z)\hat{\phi}_n(z^{-1}) f(z) \frac{dz}{2\pi iz} = \delta_{jn} \quad j = 0, 1, ..., n. \tag{2.5.2}$$

The polynomials can also be written as follows:

$$\phi_n(z) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} f_{0,0} & f_{0,1} & \dots & f_{0,n} \\ f_{1,0} & f_{1,1} & \dots & f_{1,n} \\ \dots & \dots & \dots & \\ f_{n-1,0} & f_{n-1,1} & \dots & f_{n-1,n} \\ 1 & z & \dots & z^n \end{vmatrix}, \tag{2.5.3}$$

and

$$\hat{\phi}_n(z^{-1}) = \frac{1}{\sqrt{D_n D_{n+1}}} \begin{vmatrix} f_{0,0} & f_{0,1} & \dots & f_{0,n-1} & 1 \\ f_{1,0} & f_{1,1} & \dots & f_{1,n-1} & z^{-1} \\ \dots & \dots & \dots & \dots & \dots \\ f_{n,0} & f_{n,1} & \dots & f_{n,n-1} & z^{-n} \end{vmatrix}. \tag{2.5.4}$$

Where f_{ij} are the Fourier coefficients given by

$$f_{ij} = f_{i-j} = \int_{\mathbb{T}} f(z) z^{-(i-j)} \frac{dz}{2\pi i z}. \tag{2.5.5}$$

The leading coefficient is given by

$$\chi_n = \sqrt{\frac{D_n}{D_{n+1}}}. \tag{2.5.6}$$

Remark 2.5.1. *If $f(z) \in L^1(\mathbb{T})$, then the orthogonal polynomials that satisfy (2.5.1) and (2.5.2) exist if and only if they are given by (2.5.3) and (2.5.4), as shown in Proposition 1.6.1, of [33].*

2.6 Riemann-Hilbert problems

Hilbert introduced the Riemann-Hilbert problem in his list of 23 problems around the 1900s for the purpose of proving the existence of certain linear differential equations.

The Riemann-Hilbert problem is a mathematical problem that aims to simplify the analytical factorization of a specific matrix-valued function called $V(z)$ that is defined on an oriented contour in the complex plane. Particularly, the Riemann-Hilbert problem is a boundary value problem for analytic functions with scalar or matrix values.

A typical Riemann-Hilbert problem (Σ, V) is given by an oriented contour Σ in the complex plane and a function (jump matrix) $V : \Sigma = \mathbb{T} \to GL(n, \mathbb{C})$ which consists of finding the unique matrix function $Y(z)$ that solves the following conditions:

RH-Y1 $Y(z)$ is analytic in $\mathbb{C} \setminus \Sigma$

RH-Y2 $Y_+(z) = Y_-(z)V(z), z \in \Sigma$

RH-Y3 $Y(z) \to I$, as $z \to \infty$

where, $Y_\pm(z)$ are the boundary values of $Y(z)$ defined by

$$Y_+(z) = \lim_{s \to z} f(s), \qquad \text{where } s \text{ is on the } + \text{side}$$

$$Y_-(z) = \lim_{s \to z} f(s), \qquad \text{where } s \text{ is on the } - \text{ side.}$$

In particular, we say that the $+$ side (respectively the $-$ side) lies to the left (respectively right) of the contour when on traverses it in the direction of the orientation. The problem above for $Y(z)$ is normalized because it is close to the identity as $z \to \infty$. The solution can be given by representing orthogonal polynomials as the solution of the Riemann-Hilbert problem, which will be the focus of the upcoming chapter. This fact was noticed firstly in [24], and then in [2] extended to the unit circle $\mathbb{T}$.

Definition 2.6.1. *Let γ denote a smooth curve, which may be either an arc or a closed contour. The Cauchy integral operator $C : L^p(\gamma) \to L^p(\gamma)$ can be expressed as follows:*

$$(Cf)(z) = \frac{1}{2\pi i} \int_\gamma \frac{f(\tau)}{\tau - z} d\tau \tag{2.6.1}$$

for $z \in \gamma$.

The Plemelj-Sokhotskii equations, which we will present in the following statement, play an essential part in the Riemann-Hilbert theory as they define the boundary values of Cauchy-type integrals.

Lemma 2.6.2 (Plemelj Formulas, [1])**.** *Let γ be a smooth contour. For every $f \in L^p(\gamma)$ with $1 < p < \infty$, the Cauchy type integral has the following limit values:*

$$f_\pm(t) = \pm\frac{1}{2} f(t) + \frac{1}{2\pi i} v.p \int_\gamma \frac{f(\mu)}{\mu - t} d\mu, \tag{2.6.2}$$

where $f_\pm$ denotes the limit as z approach γ along a contour entirely in the $\pm$ region respectively, and the integral $v.p \int_\gamma \frac{f(\mu)}{\mu - t} d\mu$ exists in the Cauchy principal value sense, that is the following limit exists,

$$\int_\gamma \frac{f(\mu)}{\mu - t} d\mu = v.p \int_\gamma \frac{f(\mu)}{\mu - t} d\mu = \lim_{\epsilon \to 0} \int_{\gamma \setminus B_{t,\epsilon}} \frac{f(\mu)}{\mu - t} d\mu, \tag{2.6.3}$$

where $B_{t,\epsilon} = \{z : |z - t| < \epsilon\}$.

Corollary 2.6.3 (Corollary 1, [29])**.** *If we have contour Σ and a function (jump matrix)*

$V : \Sigma = \mathbb{T} \to GL(k, \mathbb{C})$, *the additive Riemann-Hilbert problem is given by,*

RH-Y1 $Y(z)$ *is analytic in* $\mathbb{C} \setminus \Sigma$

RH-Y2 $Y_+(z) = Y_-(z) + V(z), z \in \Sigma$

RH-Y3 $Y(z) \to 0$, *as* $z \to \infty$

provides a solution that can be expressed explicitly in terms of the Cauchy integral

$$Y(z) = \frac{1}{2\pi i} \int_\Sigma \frac{V(z)}{z - s} dz. \tag{2.6.4}$$

2.7 Double-scaling limits

The double-scaling limits of Toeplitz determinants $D_n(f_t)$, with $0 \leq t < t_0$ are defined as uniform asymptotics of Toeplitz determinants which holds for $0 \leq t < t_0$ for some small t_0, as $n \to \infty$ at the same time. In both mathematics and physics, double-scaling limits are of great interest because they may explain universal behavior (see, e.g., [15]).

2.7.1 Transition from Szegő to one Fisher-Hartwig singularity

Let us have the following symbol:

$$a(z,t) = e^{V(z)}(z - e^t)^{(\alpha+\beta)}(z - e^{-t})^{(\alpha-\beta)}(z)^{(-\alpha+\beta)}e^{-i\pi(\alpha+\beta)}, \tag{2.7.1}$$

where $\alpha \pm \beta$ is non-degenerate, and $V(z)$ is analytic in a neighborhood of the unit circle. For t positive, the asymptotics of $D_n(f_t)$ are given by Szegő (see, Theorem 2.3.3), while, Theorem 2.4.2 can be used for $t = 0$, when the symbol has a Fisher-Hartwig singularity. The following asymptotic behavior of $D_n(f_t)$ is valid uniformly for $n \to \infty$ and with small $0 \leq t < t_0$.

Theorem 2.7.1 ([13]). *Let* $\alpha \in \mathbb{R}$, *with* $\alpha > -\frac{1}{2}$, $\beta \in i\mathbb{R}$. *The following asymptotic expansion holds uniformly for* $D_n(f_t(z))$ *with respect to the symbol* $f_t(z)$ *in (2.7.1) with the* *error term $o(1)$ as $n \to \infty$, and $0 \le t \le t_0$, where t_0 is sufficiently small,*

$$D_n(t) = \exp\left\{ nV_0 + nt(\alpha+\beta) \right\} \exp\left\{ \sum_{k=1}^{\infty} k \Big[V_{-k} - (\alpha-\beta)\frac{e^{-tk}}{k} \Big] \Big[V_k - (\alpha+\beta)\frac{e^{-tk}}{k} \Big] \right\} \times G_{\alpha+\beta,\alpha-\beta}\Omega(2nt)\Big(1+o(1)\Big), \tag{2.7.2}$$

where $G_{\alpha_j+\beta_j,\alpha_j-\beta_j}$ is the product of Barnes G-functions, and

$$\Omega(2nt) = \exp\left\{ \int_0^{2nt} \frac{\sigma(x)-\alpha^2+\beta^2}{x} dx + (\alpha^2-\beta^2)\log 2nt \right\}. \tag{2.7.3}$$

The function $\sigma(x)$ is real analytic on $(0,\infty)$ and exhibits the following asymptotic behavior for $x > 0$:

$$\sigma(x) = \begin{cases} \alpha^2-\beta^2+\frac{\alpha^2-\beta^2}{2\alpha}\{x - x^{1+2\alpha}C(\alpha,\beta)\}(1+\mathcal{O}(x)), & x\to 0, \quad 2\alpha \notin \mathbb{Z} \\ \alpha^2-\beta^2+\mathcal{O}(x)+\mathcal{O}(x^{1+2\alpha})+\mathcal{O}(x^{1+2\alpha}\log x), & x \to 0, \quad 2\alpha \in \mathbb{Z} \\ x^{1+2\alpha}e^{-x}\frac{1}{\Gamma(\alpha-\beta)\Gamma(\alpha+\beta)}(1+\mathcal{O}(\frac{1}{x})), & x\to\infty \end{cases}, \tag{2.7.4}$$

with

$$C(\alpha,\beta) = \frac{\Gamma(1+\alpha+\beta)\Gamma(1+\alpha-\beta)\Gamma(1-2\alpha)}{\Gamma(1-\alpha+\beta)\Gamma(1-\alpha-\beta)\Gamma(1+2\alpha)^2}\frac{1}{1+2\alpha}, \tag{2.7.5}$$

and $\Gamma(z)$ is the Euler's Γ-function.

2.7.2 Transition between one Fisher-Hartwig singularity and two Fisher-Hartwig singularities

In [34], it was studied the double scaling limit for f_t which is a symbol with one fixed Fisher-Hartwig singularity on the unit circle $\mathbb{T}$ away from 1 with parameters α_1, β_1 and one emerging Fisher-Hartwig singularity at 1 with parameters α_0, β_0. This symbol can be expressed as follows:

$$f_t(z) = e^{V(z)}z^{\beta_1}|z-z_1|g_{z_1,\beta_1}z_1^{-\beta_1}(z-e^t)^{(\alpha_0+\beta_0)}(z-e^{-t})^{(\alpha_0-\beta_0)}(z)^{(-\alpha_0+\beta_0)}e^{-i\pi(\alpha_0+\beta_0)}. \tag{2.7.6}$$

They considered two cases, $|||\beta||| < 1$ and $|||\beta||| = 1$. In the first case $|||\beta||| < 1$, the following result gives the asymptotic behavior of $D_n(f_t)$ with assuming that $\alpha_0 \in \mathbb{R}$, $\alpha_1 \in \mathbb{C}$, $\alpha_0, \Re\alpha_1 > -\frac{1}{2}$, $\beta_0 \in i\mathbb{R}$, $\beta_1 \in \mathbb{C}$. The following asymptotic expansion with the error term $o(1)$ holds true as long as $n \to \infty$ and $0 \le t \le t_0$, where t_0 is sufficiently small:

$$\begin{aligned} D_n(t) = \exp\Big\{ nV_0 + nt(\alpha_0+\beta_0) \Big\} \exp\Big\{ \sum_{k=1}^{\infty} k\Big[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big]\Big[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big] \Big\} \\ \times n^{(\alpha_1^2-\beta_1^2)} \times \exp\Big\{ -(\alpha_1-\beta_1)\sum_{k=1}^{\infty}\Big[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big] z_1^k \Big\} \\ \times \exp\Big\{ -(\alpha_1+\beta_1)\sum_{k=1}^{\infty}\Big[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big] z_1^{-k} \Big\} \times \prod_{j=0}^{1} G_{\alpha_j+\beta_j,\alpha_j-\beta_j} \\ \times \Omega(2nt)\Big(1+o(1)\Big), \end{aligned} \tag{2.7.7}$$

where $G_{\alpha_j+\beta_j,\alpha_j-\beta_j}$ and $\Omega(2nt)$ are defined in (2.4.12) and (2.7.3) respectively.

Chapter 3

Transition asymptotics with a finite number of fixed singularities

This thesis examines transition asymptotics for Toeplitz determinants with symbols f_t that have m Fisher-Hartwig singularities when $t > 0$ and $m+1$ Fisher-Hartwig singularities as a new singularity emerges at $z = 1$ when $t \to 0$. As our symbol contains at least m jump singularities, we will consider two cases. Our analysis is based on the relationship between a Riemann-Hilbert problem and the orthogonal polynomials with respect to our symbol $f_t(z)$ in (3.1.1). In this chapter, we will first investigate the case when $|||\beta^{(t)}||| < 1$, where $|||\beta^{(t)}|||$ has been defined in (2.4.10) but here with respect to the symbol $f_t(z)$ in (3.1.1). In particular, we use the differential identity to connect the Riemann-Hilbert problem and Toeplitz determinants, and also consider the case where $|||\beta^{(t)}||| = 1$ by applying concepts from [16] and [34].

3.1 The symbol

We consider a symbol $z_j = e^{i\theta_j}$, where $\theta_j \in (0, 2\pi)$, which has m Fisher-Hartwig singularities. With the exception of $z_0 = e^{i\theta_0} = 1$, where $\theta_0 = 0$, these singularities are located on the unit circle. Moreover, an additional singularity appears at $z_0 = e^{i\theta_0} = 1$ when t approaches zero.

$$f_t(z) = f(z;t) = a_t(z) \times b(z), \tag{3.1.1}$$

with

$$a_t(z) = e^{V(z)}(z-e^t)^{\alpha_0+\beta_0}(z-e^{-t})^{\alpha_0-\beta_0}z^{-\alpha_0+\beta_0}e^{-i\pi(\alpha_0+\beta_0)}$$

and

$$b(z) = z^{\sum_{j=1}^m \beta_j}\prod_{j=1}^m |z-z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j},$$

$$0 < \theta_1 < \theta_2 < \cdots < \theta_m < 2\pi.$$

When $t > 0$, $a_t(z)$ is an analytic function in the annulus $\mathbb{C} \setminus ([0, e^{-t}] \cup [e^t, \infty])$ containing the unit circle, and it has an emerging singularity with strengths α_0 and β_0 at $t = 0$. We need to compute the Fourier coefficients of $\log a(z;t)$ in order to derive the asymptotic behaviour, and the findings are as follows:

$$(\log a_t)_0 = V_0 + t(\alpha_0+\beta_0). \tag{3.1.2}$$

$$(\log a_t)_k = V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}. \tag{3.1.3}$$

$$(\log a_t)_{-k} = V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}. \tag{3.1.4}$$

We compute $a(z;t)$ Wiener-Hopf factorization, which produces,

$$\log a_t(z) = \log a_{t,+}(z) + (\log a_t(z))_0 + \log a_{t,-}(z),$$

where

$$\log a_{t,+}(z) = \sum_{k=1}^{\infty}\left(V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\right)z^k, \quad \log a_{t,-}(z) = \sum_{k=1}^{\infty}\left(V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\right)z^{-k}.$$

If $t > 0$, the following expression is produced by directly applying the Theorem 2.4.2 to the symbol $f_t(z)$ in (3.1.1):

$$\begin{aligned} D_n(f_t) &= \exp\{nV_0 + nt(\alpha_0+\beta_0)\} \times \exp\left\{\sum_{k=1}^{\infty} k\left[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\right]\left[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\right]\right\} \\ &\times n^{\sum_{j=1}^{m}(\alpha_j^2-\beta_j^2)} \times \prod_{j=1}^{m}\exp\left\{-(\alpha_j-\beta_j)\sum_{k=1}^{\infty}\big(V_k-(\alpha_0+\beta_0)\frac{e^{-tk}}{k}\big)z_j^k\right\} \\ &\times \prod_{j=1}^{m}\exp\left\{-(\alpha_j+\beta_j)\sum_{k=1}^{\infty}\big(V_{-k}-(\alpha_0-\beta_0)\frac{e^{-tk}}{k}\big)z_j^{-k}\right\} \\ &\times \prod_{1\le j<k<m}|z_j-z_k|^{2(\beta_j\beta_k-\alpha_j\alpha_k)}\left(\frac{z_k}{z_je^{i\pi}}\right)^{(\alpha_j\beta_k-\alpha_k\beta_j)} \times \prod_{j=1}^{m} G_{\alpha_j+\beta_j,\alpha_j-\beta_j}(1+o(1)), \end{aligned} \tag{3.1.5}$$

where the symbol f_t possesses m Fisher-Hartwig singularities with strengths α_j and β_j, $j = 1, 2, ...m$.

At $t = 0$, the symbol possesses $(m+1)$ singularities with α_j and β_j strengths, where $j = 0, 1, ..., m$. The following asymptotics are obtained by applying Theorem 2.4.2 again:

$$\begin{aligned} D_n(f_t) &= \exp\left\{nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k}\right\} \times \exp\left\{-(\alpha_0-\beta_0)\sum_{k=1}^{\infty}V_k - (\alpha_0+\beta_0)\sum_{k=1}^{\infty}V_{-k}\right\} \\ &\times \prod_{j=1}^{m}\exp\left\{-(\alpha_j+\beta_j)\sum_{k=1}^{\infty}V_{-k}z_j^{-k}\right\} \times \prod_{j=1}^{m}\exp\left\{-(\alpha_j-\beta_j)\sum_{k=1}^{\infty}V_kz_j^{k}\right\} \times n^{\sum_{j=0}^{m}(\alpha_j^2-\beta_j^2)} \\ &\times \prod_{0\le j<k<m}|z_j-z_k|^{2(\beta_k\beta_j-\alpha_k\alpha_j)}\left(\frac{z_k}{z_je^{i\pi}}\right)^{(\alpha_k\beta_j-\alpha_j\beta_k)} \times \prod_{j=0}^{m} G_{\alpha_j+\beta_j,\alpha_j-\beta_j}(1+o(1)). \end{aligned} \tag{3.1.6}$$

It is obvious that (3.1.5) with $t = 0$ is incompatible with (3.1.6). Therefore, it is important to find the asymptotics of Toeplitz determinants using the symbol (3.1.1) which is uniform for $0 \le t < t_0$. We will employ the Riemann-Hilbert problem analysis as in [13], [14], [16] ,[18], and [34].

3.2 Summary of results

Theorem 3.2.1. *Let $f_t(z)$ be defined as in (3.1.1), and assume that $\alpha_0 \in \mathbb{R}$, $\alpha_j \in \mathbb{C}$, $j = 1, ..., m$, where $|||\beta^{(t)}||| < 1$, and with the parameters $\alpha_0, \Re\alpha_j > -\frac{1}{2}$, $\beta_0 \in i\mathbb{R}$, $\beta_j \in \mathbb{C}$. The following asymptotic expansion is uniformly valid with an error term of $o(1)$ as n approaches infinity and $0 \le t \le t_0$, with the condition that t_0 is sufficiently small:*

$$\begin{aligned} D_n(t) = &\exp\left\{nV_0 + nt(\alpha_0+\beta_0)\right\}\exp\left\{\sum_{k=1}^{\infty} k\Big[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big]\Big[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big]\right\} \\ &\times n^{\sum_{j=1}^m(\alpha_j^2-\beta_j^2)} \times \prod_{j=1}^m \exp\left\{-(\alpha_j-\beta_j)\sum_{k=1}^{\infty}\Big[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big]z_j^k\right\} \\ &\times \prod_{j=1}^m \exp\left\{-(\alpha_j+\beta_j)\sum_{k=1}^{\infty}\Big[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\right\} \times \prod_{j=0}^m G_{\alpha_j+\beta_j,\alpha_j-\beta_j} \\ &\times \prod_{1\le j<k<m} |z_j - z_k|^{2(\beta_j\beta_k - \alpha_j\alpha_k)}\Big(\frac{z_k}{z_j e^{i\pi}}\Big)^{(\alpha_j\beta_k-\alpha_k\beta_j)}\tilde{\Omega}(2nt)\Big(1+o(1)\Big), \end{aligned} \tag{3.2.1}$$

where $G_{\alpha_j+\beta_j,\alpha_j-\beta_j}$ is the product of Barnes G-functions defined in (2.4.12),

$$\tilde{\Omega}(2nt) = \exp\Omega(2nt) = \exp\left\{\int_0^{2nt}\frac{\sigma(x)-\alpha_0^2+\beta_0^2}{x}dx + (\alpha^2-\beta_0^2)\log 2nt\right\}, \tag{3.2.2}$$

and the function $\sigma(x)$ for $x > 0$ is defined in (2.7.4).

Theorem 3.2.2. *Let $\alpha_0, \alpha_j \in \mathbb{C}$ with $\Re\alpha_0, \Re\alpha_j > -1/2$ and $\beta_0, \beta_j \in \mathbb{C}$. Suppose that $|||\beta^{(t)}||| < 1$ for $t > 0$, $|||\beta^{(t)}||| = 1$ at $t = 0$, and $\beta_{m-1} = \beta_m$ with $\Re\beta_{m-1} = \Re\beta_m = \max\{\Re\beta_j : 1 \le j \le m\}$. If t_0 is sufficiently small, then the following asymptotic expansion holds as $n \to \infty$ with a uniform error term $o(1)$:*

$$\begin{aligned} D_n(f_t(z)) = &D_n\Big(f_t(z;\alpha_0,\alpha_j,\alpha_{m-1},\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_{m-1}+1,\tilde{\beta}_m+1)\Big) \times \tilde{\Omega}(2nt)(1+o(1)) \\ &+ \prod_{j=m-1}^m z_j^{-n} \times D_n\Big(f_t(z;\alpha_0,\alpha_j,\alpha_{m-1},\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_{m-1}+1,\tilde{\beta}_m+1)\Big)K(2nt) \\ &\times\frac{n^{-2\beta_0-1}}{\Gamma(\alpha_0-\beta_0)} \times \Big(\sum_{j=m-1}^m (z_j^n - z_j^{n-1})\Big)\Sigma(t)\tilde{\Omega}(2nt)\Big(1+o(1)\Big), \end{aligned} \tag{3.2.3}$$

where $\tilde{\beta_j} = \beta_j, j = 0, ..., m-2$, $\tilde{\beta_k} = \beta_k - 1, k = m-1, m$, *and*

$$\Sigma(t) = \left(\frac{2t}{1-e^{-2t}}\right)^{\alpha_0-\beta_0} \prod_{j=m-1}^{m} (1-e^t z_j^{-1})^{\alpha_j+\tilde{\beta_j}} (1-e^{-t}z_j)^{-(\alpha_j-\tilde{\beta_j})} \exp\left\{-\sum_{k=1}^{\infty} e^{-tk} V_{-k}\right\}$$
$$\times \exp\left\{\sum_{k=1}^{\infty} e^{-tk} V_k\right\} + \left(\frac{1-e^{-2t}}{2t}\right)^{\alpha_0+\beta_0} \prod_{j=m-1}^{m} (1-e^t z_j)^{-(\alpha_j-\tilde{\beta_j})} (1-e^{-t}z_j^{-1})^{\alpha_j+\tilde{\beta_j}}$$
$$\times \exp\left\{-\sum_{k=1}^{\infty} e^{-tk} V_k\right\} \exp\left\{-\sum_{k=1}^{\infty} e^{tk} V_{-k}\right\}.$$

Remark 3.2.3. *In addition to the trivial Fisher-Hartwig representation, we can define the non-trivial Fisher-Hartwig representation of the symbol* $f_t(z)$ *at* $t = 0$ *as* $\hat{\beta}_0 = \beta_0 + n_0$ *and* $\hat{\beta}_m = \beta_m - n_1$, *where* $\beta_0 = \min \Re\beta_j$ *and* $\beta_m = \max \Re\beta_j$. *In this case, we have*

$$D_n(f_t(z)) = R\bigg(f_t(z; \beta_0, \beta_j, \beta_m)\bigg) \times \tilde{\Omega}(2nt)(1+o(1)) + (z_m^{-n})^n \times n^{-2\hat{\beta}_j - 1} \times \frac{\Gamma(1+\alpha_j+\hat{\beta}_j)}{\Gamma(\alpha_j - \hat{\beta}_j)}$$
$$\times V_j \times \tilde{\Omega}(2nt) \times \frac{K(2nt)}{e^{nt}} \times \frac{n^{-2\beta_0-1}}{\Gamma(1+\alpha_0+\beta_0)} \times (1-e^{-2t})^{-2\beta_0-1}\Sigma'(t)$$
$$\times R\bigg(f_t(z; \hat{\beta}_0, \hat{\beta}_j, \hat{\beta}_m)\bigg)\bigg(1+o(1)\bigg), \tag{3.2.4}$$

where $R\bigg(f(z; \hat{\beta}_0, \hat{\beta}_j, \hat{\beta}_m)\bigg)$ *corresponds to the RHS of (3.2.1) for symbol* f_t *with* $\hat{\beta}$*'s parameters and without the error term or* $\tilde{\Omega}(2nt)$,

$$\Sigma'(t) = \Bigg[\left(\frac{z_m - e^t}{z_m - e^{-t}}\right)^{\alpha_m + \hat{\beta_m}} \exp\left\{2\sum_{k=1}^{\infty} V_k(\sinh(tk))\right\} \left(\frac{2t}{1-e^{-2t}}\right)^{\alpha_0-\beta_0}$$
$$\times (1-e^{-t}z_j)^{\sum_{j=1}^{m-1} -(\alpha_j - \hat{\beta_j})} \times (1-e^{-t}z_j^{-1})^{\sum_{j=1}^{m-1} -(\alpha_m + \hat{\beta_m})}$$
$$+ \left(\frac{z_m - e^t}{z_m - e^{-t}}\right)^{\alpha_m - \hat{\beta_m}} \exp\left\{-2\sum_{k=1}^{\infty} V_{-k}(\sinh(tk))\right\} \times \left(\frac{2t}{1-e^{-2t}}\right)^{-(\alpha_0+\beta_0)} \tag{3.2.5}$$
$$\times (1-e^{-t}z_m)^{\sum_{j=1}^{m-1}(\alpha_m - \hat{\beta_m})} \times (1-e^{-t}z_m^{-1})^{\sum_{j=1}^{m-1}(\alpha_m + \hat{\beta_m})}\Bigg].$$

Remark 3.2.4. *Let* $\alpha_0, \alpha_j > -1/2, \beta_0, \beta_j \in \mathbb{C}$, *and* $|||\beta^{(t)}||| = 1$ *for* $t \geq 0$. *Then the Toeplitz determinant* $D_n(f_t)$ *generated by the symbol* f_t *in (3.1.1) has two possible asymptotics depending on the position of* $\Re\beta_0$.

1. *If* $\min\{\Re\beta_j : 1 \le j \le m\} < \Re\beta_0 < \max\{\Re\beta_j :: 1 \le j \le m\}$*, then the asymptotics is given by Theorem 1.13 of [16].*

2. *If* $\Re\beta_0 = \min\{\Re\beta_j : 1 \le j \le m\}$ *or* $\Re\beta_0 = \max\{\Re\beta_j : 1 \le j \le m\}$*, then the asymptotics is given by Theorem 3.2.2.*

3.3 The Riemann-Hilbert problem formulation of orthogonal polynomials

Now we show how to formulate the orthogonal polynomial problem as Riemann-Hilbert problem, building on the work of [24]. We consider the following 2×2 matrix-valued function $Y^{(n)}(z) \equiv Y(z)$, $n \ge n_0$:

$$Y(z) = \begin{pmatrix} \chi_n^{-1}\phi_n(z) & \chi_n^{-1}\int_T \frac{\phi_n(\xi)}{\xi - z}\frac{f_t(\xi)d\xi}{2\pi i\xi^n} \\ -\chi_{n-1}z^{n-1}\hat{\phi}_{n-1}(z^{-1}) & -\chi_{n-1}\int_T \frac{\hat{\phi}_{n-1}(\xi^{-1})}{\xi - z}\frac{f_t(\xi)d\xi}{2\pi i\xi^n} \end{pmatrix}. \tag{3.3.1}$$

Notice that $Y(z)$ is the only solution to the following Riemann-Hilbert problem (see, e.g., [16]):

RH-Y1: $Y : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic.

RH-Y2: For $z \in \mathbb{T} \setminus \cup_{j=1}^{m} z_j$, where $j = 1, ..., m$, $Y(z)$ has continuous boundary values $Y_+(z)$ and $Y_-(z)$, related by the jump condition

$$Y_+(z) = Y_-(z)\begin{pmatrix} 1 & z^{-n}f_t(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{T}. \tag{3.3.2}$$

RH-Y3: The asymptotic behaviour for $Y(z)$ as $z \to \infty$ is given by

$$Y(z) = (I + \mathcal{O}(\frac{1}{z}))\begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{3.3.3}$$

RH-Y4: As $z \to z_j$, we have

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_j|)^{2\alpha_j} \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_j|)^{2\alpha_j} \end{pmatrix}, \qquad \text{if} \quad \alpha_j \neq 0 \tag{3.3.4}$$

and

$$Y(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z - z_j|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z - z_j|) \end{pmatrix}, \qquad \text{if} \quad \alpha_j = 0, \quad \beta_j \neq 0. \tag{3.3.5}$$

The following proposition demonstrates how the previous Riemann-Hilbert problem relates to the orthogonal polynomials (2.5.1) and (2.5.2).

Proposition 3.3.1. *If the Riemann-Hilbert problem stated above is solved by $Y(z)$ in* (3.3.1)*, then the polynomials $\phi_n(z)$ and $\hat{\phi}_n(z)$ satisfy the orthogonal conditions* (2.5.1) *and* (2.5.2)*.*

Proof. The solution of the Riemann-Hilbert problem is denoted by $Y(z)$, and each entry in the matrix is examined separately. Using the RH-Y3 condition, we can also observe the asymptotic behaviour of each entry as $z \to \infty$.

$$Y(z) = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \\ Y_{21}(z) & Y_{22}(z) \end{pmatrix} = \begin{pmatrix} z^n + \mathcal{O}(z^{n-1}) & \mathcal{O}(z^{-n-1}) \\ \mathcal{O}(z^{n-1}) & \mathcal{O}(z^{-n}) + \mathcal{O}(z^{-n-1}) \end{pmatrix}. \tag{3.3.6}$$

Assuming $n \geq 1$, we will look at the first row of the jump condition in RH-Y2,

$$\begin{pmatrix} Y_{11}(z) & Y_{12}(z) \end{pmatrix}_+ = \begin{pmatrix} Y_{11}(z) & Y_{12}(z) \end{pmatrix}_- \begin{pmatrix} 1 & z^{-n} f_t(z) \\ 0 & 1 \end{pmatrix}.$$

When we match the values of the row vector, we get:

1. $(Y_{11})_+(z) = (Y_{11})_-(z)$
2. $(Y_{12})_+(z) = (Y_{11})_-(z) z^{-n} f_t(z) + (Y_{12})_-(z)$

We may observe from RH-Y3 (3.3.6) that $Y_{11}(z) = z^n + \mathcal{O}(z^{n-1})$ is a monic polynomial of degree n. Let us denote $p_n(z) = Y_{11}(z)$. The second row is

$$(Y_{12})_+(z) = (Y_{12})_-(z) + p_n(z)z^{-n}f_t(z),$$

which is an additive Riemann-Hilbert problem. The answer, then, is as follows based on the Plemelj formula:

$$(Y_{12})(z) = \int_{\mathbb{T}} \frac{p_n(s)}{s-z}\frac{f_t(s)ds}{s^n 2\pi i}.$$

Expanding $\frac{1}{s-z}$ and applying the asymptotics at infinity $(Y_{12})(z) = \mathcal{O}(z^{-n-1})$ as $z \to \infty$, yields the following result:

$$\begin{aligned}(Y_{12})(z) &= -\frac{1}{z}\int_{\mathbb{T}} \frac{p_n(s)s^{-n}f_t(s)}{1-s/z}\frac{ds}{2\pi i} \\ &= -\sum_{j=0}^{\infty} z^{-j-1}\int_{\mathbb{T}} p_n(s)s^{-n+j}f_t(s)\frac{ds}{2\pi i} \\ &= -\sum_{j=0}^{\infty} z^{-j-1}\int_{\mathbb{T}} p_n(s)s^{-n+j+1}f_t(s)\frac{ds}{2\pi i s}.\end{aligned}$$

Since $Y_{12} = \mathcal{O}(z^{-n-1})$, we have

$$\int_{\mathbb{T}} p_n(s)s^{-n+j+1}\frac{f_t(s)ds}{2\pi i s} = \begin{cases} 0 \quad \text{If} \quad j = 0, ..., n-1, \\ \chi_n^{-1} \quad \text{If} \quad j = n. \end{cases}$$

Notice that $0 \le j \le n-1$, implies that $-(n-1) \le -n+j+1 \le 0$, and therefore, $p_n(s)$ satisfies

$$\int_{\mathbb{T}} p_n(s)s^{-k}\frac{f_t(s)ds}{2\pi i s} = \begin{cases} 0 \quad \text{If} \quad k = 0, ..., n-1, \\ \chi_n^{-1} \quad \text{If} \quad k = n. \end{cases}$$

Consequently, we have unique polynomials $p_n(z) = \chi_n^{-1}\phi_n(z)$.

For the second column, we have

$$\begin{pmatrix} Y_{21}(z) & Y_{22}(z) \end{pmatrix}_+ = \begin{pmatrix} Y_{21}(z) & Y_{22}(z) \end{pmatrix}_- \begin{pmatrix} 1 & z^{-n}f_t(z) \\ 0 & 1 \end{pmatrix}.$$

This yields the following

1. $(Y_{21})_+(z) = (Y_{21})_-(z)$

2. $(Y_{22})_+(z) = (Y_{21})_-(z)z^{-n}f_t(z) + (Y_{22})_-(z)$

Again we observe that $Y_{21}(z)$ is analytic in the complex plane from the first of the preceding identities. Additionally, we observe that it is a polynomial of degree $n-1$, denoted by p_{n-1}, using the asymptotics condition (RH-Y3). From the second identity above and the Plemelj formula, we find that

$$Y_{22}(z) = \int_{\mathbb{T}} \frac{Y_{21}^{(n-1)}(s)}{s-z} \frac{f_t(s)ds}{2\pi i s^n}.$$

Also, we deduce that

$$Y_{22}(z) = z^{-n} + \mathcal{O}(z^{-(n+1)}) \quad \text{as} \quad z \to \infty$$

(see (RH-Y3)). As before, expanding $\frac{1}{s-z}$, we obtain

$$\begin{aligned} Y_{22}(z) &= -\frac{1}{z}\int_{\mathbb{T}} \frac{p_{n-1}(s)s^{-n}f_t(s)}{1-s/z}\frac{ds}{2\pi i} \\ &= -\sum_{j=0}^{\infty} z^{-j-1}\int_{\mathbb{T}} p_{n-1}(s)s^{-n+j}f_t(s)\frac{ds}{2\pi i} \\ &= -\sum_{j=0}^{\infty} z^{-j-1}\int_{\mathbb{T}} p_{n-1}(s)s^{-n+j+1}f_t(s)\frac{ds}{2\pi i s}. \end{aligned}$$

Thus, we have

$$\int_{\mathbb{T}} p_{n-1}(s)s^{-n+j+1}f_t(s)\frac{ds}{2\pi i s} \begin{cases} = -1 & \text{if} \quad j = n-1 \\ = 0 & \text{if} \quad 0 \le j \le n-2 \end{cases}$$

Note that $0 \le j \le n-2$ is equivalent to $-(n-1) \le -n+j+1 \le -1$. Let $p_{n-1}(s) = -\chi_{n-1}z^{n-1}\hat{p}_{n-1}(z^{-1})$. Then

$$-\chi_{n-1}\int_{\mathbb{T}} \hat{p}_{n-1}(s^{-1})s^{n-1-n+j+1}f_t(s)\frac{ds}{2\pi i s} = -\chi_{n-1}\int_{\mathbb{T}} \hat{p}_{n-1}(s^{-1})s^{j}f_t(s)\frac{ds}{2\pi i s} = 0$$

for $0 \leq j \leq n-2$. Therefore, for $0 \leq j \leq n-2$, we have

$$\int_{\mathbb{T}} p_{n-1}(s) s^{-n+j+1} f_t(s) \frac{ds}{2\pi i s} = 0.$$

Furthermore, in the case when $j = n-1$, we obtain

$$\int_{\mathbb{T}} p_{n-1}(s) s^{-n+n-1+1} f_t(s) \frac{ds}{2\pi i s} = \int_{\mathbb{T}} p_{n-1}(s) f_t(s) \frac{ds}{2\pi i s}$$
$$= -\chi_{n-1} \int_{\mathbb{T}} \hat{p}_{n-1}(s^{-1}) s^{n-1} f_t(s) \frac{ds}{2\pi i s} = -1.$$

Now, we will show that the solution given in (3.3.1) is unique . We first assume that

$$\det Y(z) = 1 \quad z \in \mathbb{C} \setminus \mathbb{T}.$$

By utilising (3.3.2), we have

$$(\det Y)_+(z) = \det(Y_+(z)) = \det(Y_-(z)). \det \begin{pmatrix} 1 & z^{-n} f_t(z) \\ 0 & 1 \end{pmatrix}$$
$$= (\det Y)_-(z).$$

As a result, by applying (3.3.3), we get

$$\det Y(z) = 1 + \mathcal{O}(1/z) \quad \text{as} \quad z \to \infty.$$

Because of this, $\det Y(z)$ is bounded and entire, and since $\det Y(z) = 1$, by Liouville's theorem, it is constant. The matrix-valued function $Y(z)$ is invertible since its determinant equals one. Next we will examine the matrix $X(z)$ defined by

$$X(z) = \tilde{Y}(z) Y^{-1}(z), \quad z \in \mathbb{C} \setminus \mathbb{T}.$$

Observe that the function $X(z)$ is well defined and analytic on $\mathbb{C} \setminus \mathbb{T}$. Further, for $z \in \mathbb{T}$,

$$\begin{aligned} X_+(z) &= \tilde{Y}_+(z) Y_+^{-1}(z) \\ &= \tilde{Y}_-(z) \begin{pmatrix} 1 & z^{-n} f_t(z) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z^{-n} f_t(z) \\ 0 & 1 \end{pmatrix}^{-1} Y_-^{-1}(z) \\ &= X_-(z). \end{aligned}$$

Consequently, $X(z)$ is an entire function. We obtain the following for $z \to \infty$:

$$\begin{aligned} X(z) &= \tilde{Y}(z) Y^{-1}(z) \\ &= (I + \mathcal{O}(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix} \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}^{-1} (I + \mathcal{O}(1/z))^{-1} \\ &= (I + \mathcal{O}(1/z)). \end{aligned}$$

Once again, we apply Liouville's theorem to get $X(z) = I$, which implies $\tilde{Y}(z) = Y(z)$. Thus, the solution in (3.3.1) is unique. □

3.4 The Nonlinear steepest descent method

In this section, we will analyze the solution of the Riemann-Hilbert problem for $Y(z)$ in (3.3.1) for large n. The Riemann-Hilbert problem must endure a series of reversible transformations in order to evaluate the asymptotic behaviour of its solution. In the 1990s, Deift and Zhou introduced the method of nonlinear steepest descent (see [21]), which is one of the classical techniques that can provide an asymptotically complete expansion for OPs. It includes a sequence of transformations, each of which further simplifies the original Riemann-Hilbert problem and brings us closer to the solution of $Y(z)$. The goal is to discover a solution to the Riemann-Hilbert problem $R(z)$ that is close to the identity and whose jump matrix behaves as the identity matrix; this is known as the small norm Riemann-Hilbert problem. Then, we can reverse the transformations to obtain an answer to the original problem involving $Y(z)$. We address the analysis in a manner analogous to [13], [16], [18], and [34].

The final transformation generates a Riemann-Hilbert problem with a small norm for the function $R(z)$, which can be expressed in the following manner:

The Riemann-Hilbert problem for R

(RH-R1): It is analytic in $\mathbb{C} \setminus \Sigma_R \to \mathbb{C}^{2\times 2}$.

(RH-R2): It satisfies the following jump conditions,

$$R_+(z) = R_-(z) J_R(z) \quad z \in \Sigma_R. \tag{3.4.1}$$

(RH-R3): As infinity, the function behaves as follows:

$$R(z) = I + \mathcal{O}(z^{-1}) \quad \text{as} \quad z \to \infty. \tag{3.4.2}$$

According to the standard theory for small-norm Riemann-Hilbert problems [19] and [20], $R(z)$ exists for n large and has a uniform behaviour for $z \in \mathbb{C} \setminus \Sigma_R$,

$$R(z) = I + O(n^{-1}).$$

3.4.1 First transformation $T(z)$ (normalization)

To fix the behaviour of $Y(z)$ and to normalize the Riemann-Hilbert problems at infinity, we define the first transformation $T(z)$ as follows:

$$T(z) = Y(z) \begin{cases} z^{-n\sigma_3} & : |z| > 1 \\ I & : |z| < 1 \end{cases} \tag{3.4.3}$$

where σ_3 is the Pauli matrix defined by

$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{3.4.4}$$

The function $T(z)$ in (3.4.3) solves the following Riemann-Hilbert problem:

RH-T1: It is analytic for $z \in \mathbb{C} \setminus \mathbb{T}$.

RH-T2: It has the following jump condition at $z \in \mathbb{T}$,

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & f_t(z) \\ 0 & z^{-n} \end{pmatrix} .done \tag{3.4.5}$$

RH-T3: At infinity, it behaves as follows:

$$T(z) = \big(I + \mathcal{O}(z^{-1})\big).$$

RH-T4 : The asymptotic behaviour as $z \to z_j, \quad j = 1, ..., m$, remains the same as that of $Y(z)$ in (3.3.4) and (3.3.5).

Observe that the Riemann-Hilbert problems for $Y(z)$ and $T(z)$ are equivalent, so if we can solve one, we can solve the other using straightforward algebraic manipulation. Consequently, the Riemann-Hilbert problem for $T(z)$ has only one solution.

3.4.2 Second transformation $S(z)$

This step (known as the opening of the lenses) will deform the unit circle as shown in Figure 3.1. Since the jump matrix $V_T(z)$ in (RH-T2) has an absolute value of 1, oscillations happen quickly when n is large. So, we will come up with a new transformation that makes this behaviour on $\mathbb{T}$ exponentially small. Now, let us set up a new function as follows:

$$S(z) = \begin{cases} T(z), & \text{for } z \text{ outside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ f_t(z)^{-1} z^{-n} & 1 \end{pmatrix}, & \text{for } |z| > 1 \quad \text{and inside the lenses,} \\ T(z) \begin{pmatrix} 1 & 0 \\ -f_t(z)^{-1} z^{n} & 1 \end{pmatrix}, & \text{for} |z| < 1 \quad \text{and inside the lenses} \end{cases} \tag{3.4.6}$$

where the jump matrix $V_T(z)$ is factorized,

$$\begin{aligned} V_T(z) &= \begin{pmatrix} 1 & 0 \\ z^{-n} f_t(z)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & f_t(z) \\ -f_t(z)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{n} f_t(z)^{-1} & 1 \end{pmatrix} \\ &= V_1(z) V_N(z) V_2(z). \end{aligned} \tag{3.4.7}$$

This factorization has specific characteristics. The matrix-valued function $f_t(z)$ has analytic continuation on the outside and inside of the unit disc. Observe that the off-diagonal terms of V_1 and V_2 decay as expected.

Then the following Riemann-Hilbert problem can be solved using the function $S(z)$:

RH-S1: $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$, where $\Sigma = \cup_{j=0}^{m} (\Sigma_j \cup \Sigma_j' \cup \Sigma_j'')$.

RH-S2: The boundary values are defined by the jump conditions

$$S_+(z) = S_-(z) \begin{pmatrix} 1 & 0 \\ f_t(z)^{-1} z^{\pm n} & 1 \end{pmatrix} \quad \text{for } z \in \cup_{j=0}^{m} (\Sigma_j \cup \Sigma_j''),$$

with the minus sign in the exponent on Σ_j, and plus sign on Σ_j'',

$$S_+(z) = S_-(z) \begin{pmatrix} 0 & f_t(z) \\ -f_t(z)^{-1} & 0, \end{pmatrix} \quad \text{for } z \in \cup_{j=0}^{m} \Sigma_j'.$$

RH-S3: $S(z) = \big(I + \mathcal{O}(z^{-1})\big)$ as $z \to \infty$.

RH-S4: As $z \to z_j$ $(j = 1, ..., m)$ and $z \in \mathbb{C} \setminus \mathbb{T}$ remains outside of the lenses,

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|z - z_j|) \\ \mathcal{O}(1) & \mathcal{O}(\log|z - z_j|) \end{pmatrix}, \quad \alpha_j = 0, \quad \beta_j \neq 0 \tag{3.4.8}$$

and

$$S(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_j|)^{2\alpha_j} \\ \mathcal{O}(1) & \mathcal{O}(1) + \mathcal{O}(|z - z_j|)^{2\alpha_j} \end{pmatrix}, \quad \alpha_j \neq 0. \tag{3.4.9}$$

The asymptotic behaviour for $S(z)$ as $z \to z_j$ in other sectors may be determined by ap-

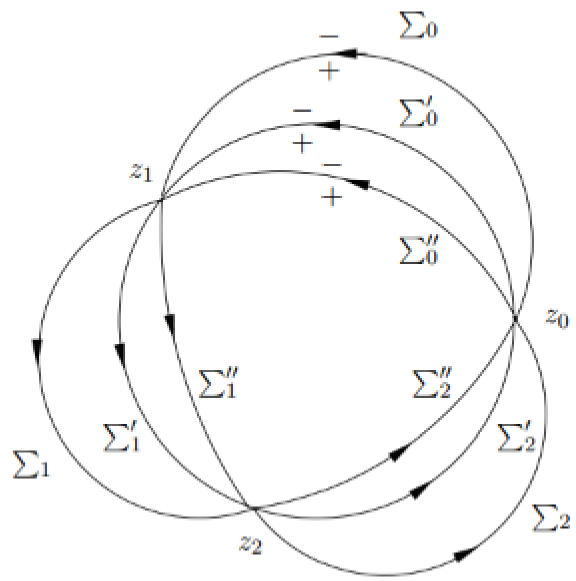

Figure 3.1: $S(z)$ RH problems

plying suitable jump conditions. The Riemann-Hilbert problem $(S(z), \Sigma)$ is referred to as a deformation of the problem $(T(z), \mathbb{T})$.

Finding good approximations to $S(z)$ in various regions of the complex plane is the subsequent step of the steepest descent analysis. Let us begin by encircling each of the points z_j $(j = 0, 1, ..., m)$ by the sufficiently small discs

$$U_{z_j} = \{z : |z - z_j| < \epsilon\}. \tag{3.4.10}$$

By ignoring the jumps in $\cup_{j=0}^{m}(\Sigma_j \cup \Sigma_j'')$ that tend to the identity matrix and are away from $U_{z_j}, j = 0, ..., m$, we are left with the Riemann-Hilbert problem that is independent of n and whose solution is a good approximation of $S(z)$ away from the singularities z_j.

3.4.3 Global parametrix $N(z)$

We consider the following problem with the parametrix $N(z)$:

RH-N1: It is analytic for $z \in \mathbb{C} \setminus \mathbb{T}$.

RH-N2: $N_+(z) = N_-(z) \begin{pmatrix} 0 & f_t(z) \\ -f_t(z)^{-1} & 0 \end{pmatrix}$ for $z \in \mathbb{T} \setminus \cup_{j=0}^{m} z_j$.

RH-N3: As $z \to \infty$, the function has the following behaviour: $N(z) = (I + \mathcal{O}(z^{-1}))$.

The function $D(z)$ can be used to solve this Riemann-Hilbert problem by ignoring $\cup_{j=0}^{m} U_{z_j}$ and $\cup_{j=0}^{1}(\Sigma_j \cup \Sigma_j'')$. The following represents the function $N(z)$

$$N(z) = \begin{cases} D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & : |z| < 1 \\ D(z)^{\sigma_3} & : |z| > 1 \end{cases} \tag{3.4.11}$$

where $D(z)$ is the Szegő function associated with $f_t(z)$ defined by

$$D(z) = \exp\Big\{\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log f_t(\tau)}{\tau - z} d\tau\Big\}. \tag{3.4.12}$$

Notice that $D(z)$ is analytic in $\mathbb{C} \setminus \mathbb{T}$ and has the jumps

$$D_+(z) = D_-(z) f_t(z), \quad z \in \mathbb{T} \setminus \cup_{j=0}^{m} z_j.$$

Similarly to [13], [16], and [34], we can explicitly solve this problem by utilizing the Szegő function $D(z)$.

Lemma 3.4.1. *By figuring out the integral with respect to our symbol $f_t(z)$ in (3.1.1), we obtain the following simple formula for the function $D(z)$:*

$$D(z) = \begin{cases} \prod_{j=1}^{m} (\frac{z-z_j}{z_j e^{i\pi}})^{\alpha_j+\beta_j} (z-e^t)^{\alpha_0+\beta_0} e^{-i\pi(\alpha_0+\beta_0)} \exp\{\sum_{k=0}^{\infty} V_k z^k\}, & |z| < 1 \\ \prod_{j=1}^{m} (\frac{z-z_j}{z})^{-\alpha_j+\beta_j} (z-e^{-t})^{-\alpha_0+\beta_0} z^{\alpha_0-\beta_0} \exp\{\sum_{k=1}^{\infty} V_{-k} z^{-k}\}, & |z| > 1 \end{cases} \tag{3.4.13}$$

Proof. We will first calculate the $\log f_t(\tau)$, following the procedure described in [34]:

$$\begin{aligned} \log f_t(\tau) = {} & V(\tau) + \sum_{j=1}^{m} \beta_j \log \tau - \sum_{j=1}^{m} \beta_j \log z_j + \sum_{j=1}^{m} 2\alpha_j \log|\tau - z_j| \\ & + \sum_{j=1}^{m} \log g_{z_j,\beta_j}(\tau) + (\alpha_0+\beta_0)\log(\tau - e^t) + (\alpha_0 - \beta_0)\log(\tau - e^{-t}) \\ & - (\alpha_0 - \beta_0)\log\tau - i\pi(\alpha_0+\beta_0). \end{aligned} \tag{3.4.14}$$

The first function, $V(\tau)$, is analytic. We can expand $\frac{1}{\tau - z} = \frac{1}{\tau} + \frac{z}{\tau^2} + \frac{z^2}{\tau^3} + ...$, for $|z| < 1$, and by applying the residual theorem, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{V(\tau)}{\tau - z} d\tau &= \frac{1}{2\pi i}\int_{\mathbb{T}} \left\{ ... + V_{-1}\tau^{-1} + V_0 + V_1\tau + ... \right\}\left\{ \frac{1}{\tau} + \frac{z}{\tau^2} + ... \right\} d\tau \\ &= \sum_{k=0}^{\infty} V_k z^k.\end{aligned}$$

Expanding $\frac{1}{\tau - z} = -\frac{1}{z} - \frac{\tau}{z^2} - \frac{\tau^2}{z^3} - ...$ for $|z| > 1$ yields the following result,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{V(\tau)}{\tau - z} d\tau &= \frac{1}{2\pi i}\int_{\mathbb{T}} \left\{ ... + V_{-1}\tau^{-1} + V_0 + V_1\tau + ... \right\}\left\{ -\frac{1}{z} - \frac{\tau}{z^2} - ... \right\} d\tau \\ &= -\sum_{k=1}^{\infty} V_{-k} z^{-k}.\end{aligned}$$

We apply integration by parts, the substitution $\tau = e^{i\theta}$, and the residue theorem to obtain the following:

$$\begin{aligned}\frac{\sum_{j=1}^m \beta_j}{2\pi i}\int_{\mathbb{T}} \frac{\log\tau}{\tau - z} d\tau &= \frac{\sum_{j=1}^m \beta_j}{2\pi i}\int_{\mathbb{T}} \frac{\log(e^{i\theta}) i e^{i\theta}}{e^{i\theta} - z} d\theta \\ &= \frac{\sum_{j=1}^m \beta_j}{2\pi i}\Big[i\theta \log(e^{i\theta} - z) \Big]_0^{2\pi} - \frac{\sum_{j=1}^m \beta_j}{2\pi i}\int_0^{2\pi} i\log(e^{i\theta} - z) d\theta \\ &= \sum_{j=1}^m \beta_j \Big(\log(1 - z) \Big) - \frac{\sum_{j=1}^m \beta_j}{2\pi i}\int_{\mathbb{T}} \frac{\log(\tau - z)}{\tau} d\tau.\end{aligned} \tag{3.4.15}$$

For $|z| < 1$, we have

$$\begin{aligned}\frac{\sum_{j=1}^m \beta_j}{2\pi i}\int_{\mathbb{T}} \frac{\log\tau}{\tau - z} d\tau &= \sum_{j=1}^m \beta_j \Bigg(\log(1 - z) - \frac{1}{2\pi i}\int_{\mathbb{T}} \left\{ \log\tau - \frac{z}{\tau} - \frac{z^2}{2\tau^2} - ... \right\}\left\{ \frac{1}{\tau} \right\} d\tau \Bigg) \\ &= \sum_{j=1}^m \beta_j \Bigg(\log(1 - z) - \frac{1}{2\pi i}\int_0^{2\pi} i\log(e^{i\theta}) d\theta - 0 \Bigg) \\ &= \sum_{j=1}^m \beta_j \Big(\log(1 - z) - i\pi \Big),\end{aligned} \tag{3.4.16}$$

and for $|z| > 1$

$$\frac{\sum_{j=1}^{m}\beta_j}{2\pi i}\int_{\mathbb{T}}\frac{\log\tau}{\tau - z}d\tau = \sum_{j=1}^{m}\beta_j\Bigg(i\pi + \log(z-1) - \frac{1}{2\pi i}\int_{\mathbb{T}}\Bigg\{i\pi + \log z - \frac{\tau}{z} - \frac{\tau^2}{2z^2} - ...\Bigg\}\Bigg\{\frac{1}{\tau}\Bigg\}d\tau\Bigg)$$
$$= \sum_{j=1}^{m}\beta_j\Bigg(\log(z-1) - \log z\Bigg). \tag{3.4.17}$$

Combining the residue theorem with the above expansion, we get:

$$\frac{-\sum_{j=1}^{m}\beta_j}{2\pi i}\int_{\mathbb{T}}\frac{\log z_j}{\tau - z}d\tau = \begin{cases} -\sum_{j=1}^{m}\beta_j\log z_j, & |z| < 1, \\ 0, & |z| > 1. \end{cases} \tag{3.4.18}$$

Now, we define h_{α_j} by setting

$$h_{\alpha_j}(z) = |z - z_j|^{\alpha_j} = (z - z_j)^{\alpha_j/2}(z^{-1} - z_j^{-1})^{\alpha_j/2} = \frac{(z - z_j)^{\alpha_j}}{(zz_je^{il_j})^{\alpha_j/2}}, \tag{3.4.19}$$

where

$$l_j = \begin{cases} 3\pi, & \text{for} \quad 0 < \theta < \theta_j \\ \pi, & \text{for} \quad \theta_j < \theta < 2\pi \end{cases} \tag{3.4.20}$$

and $z = e^{i\theta}, z_j = e^{i\theta_j}, \theta_j \neq 0,\ 0 \leq \theta < 2\pi$. We then write

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}2\alpha_j\log|\tau - z_j|}{\tau - z}d\tau = \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}2\alpha_j\log(\tau - z_j) - \sum_{j=1}^{m}\alpha_j\log(\tau z_j e^{il_j})}{\tau - z}d\tau$$
$$= \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}2\alpha_j\log(\tau - z_j)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}\alpha_j\log(\tau)}{\tau - z}d\tau$$
$$- \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}\alpha_j\log(z_j)}{\tau - z}d\tau - \frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^{m}\alpha_j\log(e^{il_j})}{\tau - z}d\tau. \tag{3.4.21}$$

We can evaluate the first term in (3.4.21) and the expansion that follows by using the residue theorem as follows:

$$\log(\tau - z_j) = \log(\exp(i\pi)) + \log(z_j) - \sum_{k=1}^{\infty} \frac{z_j^{-k}\tau^k}{k}. \tag{3.4.22}$$

Recall that,

$$\log(1-z) = -\sum_{n=1}^{\infty} \frac{z^n}{n!}, \quad \text{for} \quad |z| < 1. \tag{3.4.23}$$

Consequently,

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\sum_{j=1}^{m} 2\alpha_j \log(\tau - z_j)}{\tau - z} d\tau = \begin{cases} \sum_{j=1}^{m} 2\alpha_j \log(z - z_j), & |z| < 1, \\ 0, & |z| > 1. \end{cases} \tag{3.4.24}$$

The second term was evaluated in (3.4.16) and (3.4.17), and the third one in (3.4.18). In order to determine the final term, we note that

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\sum_{j=1}^{m} \alpha_j \log(e^{il_j})}{\tau - z} d\tau = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\alpha_j \log(e^{2\pi i})}{\tau - z} d\tau + \frac{\alpha_j}{\beta_j}\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log g_{z_j,\beta_j}(\tau)}{\tau - z} d\tau, \tag{3.4.25}$$

where

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log g_{z_j,\beta_j}(\tau)}{\tau - z} d\tau &= \frac{1}{2\pi i}\int_{0}^{\theta_j} \frac{i\pi\beta_j}{\tau - z} d\tau - \frac{1}{2\pi i}\int_{\theta_j}^{2\pi} \frac{i\pi\beta_j}{\tau - z} d\tau \\ &= \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{i\pi\beta_j}{\tau - z} d\tau - 2\frac{1}{2\pi i}\int_{\theta_j}^{2\pi} \frac{i\pi\beta_j}{\tau - z} d\tau. \end{aligned}$$

By (3.4.18), we get

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{\log g_{z_j,\beta_j}(\tau)}{\tau - z} ds = \begin{cases} i\pi\beta_j - 2\frac{1}{2\pi i}\int_{\theta_j}^{2\pi} \frac{i\pi\beta_j}{\tau - z} d\tau, & \text{for} |z| < 1 \\ -2\frac{1}{2\pi i}\int_{\theta_j}^{2\pi} \frac{i\pi\beta_j}{\tau - z} d\tau, & \text{for} |z| > 1 \end{cases} \tag{3.4.26}$$

and

$$2\frac{1}{2\pi i}\int_{\theta_j}^{2\pi} \frac{i\pi\beta_j}{\tau - z} d\tau = \beta_j\big(\log(1-z) - \log(z - z_j) - i\pi\big).$$

In this case, we have

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\log g_{z_j,\beta_j}(\tau)}{\tau-z}d\tau=\begin{cases}\beta_j\log(z-z_j)-\beta_j\log(1-z), & \text{for}\quad |z|<1,\\ \beta_j\log(z-z_j)-\beta_j\log(z-1), & \text{for}\quad |z|>1.\end{cases}\tag{3.4.27}$$

Thus, by combining (3.4.18) and (3.4.27), we obtain

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\alpha_j\log(e^{il_j})}{\tau-z}d\tau=\begin{cases}\alpha_j\big(\log(z-z_j)-\log(1-z)\big), & \text{for}\quad |z|<1,\\ \alpha_j\big(\log(z-z_j)-\log(z-1)\big), & \text{for}\quad |z|>1.\end{cases}\tag{3.4.28}$$

Therefore, we have for $|z|<1$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^m 2\alpha_j\log|\tau-z_j|}{\tau-z}d\tau&=\sum_{j=1}^m\Bigg(2\alpha_j\log(z-z_j)-\alpha_j\log(1-z)-i\pi\alpha_j-\alpha_j\log z_j\\&\quad-\alpha_j\log(z-z_j)+\alpha_j\log(1-z)\Bigg)\\&=\sum_{j=1}^m\Bigg(\alpha_j\log(z-z_j)-\alpha_j\log z_j-i\pi\alpha_j\Bigg),\end{aligned}\tag{3.4.29}$$

and for $|z|>1$,

$$\frac{1}{2\pi i}\int_{\mathbb{T}}\frac{\sum_{j=1}^m 2\alpha_j\log|\tau-z_j|}{\tau-z}d\tau=\alpha_j\log z-\alpha_j\log(z-z_j).\tag{3.4.30}$$

Using the residue theorem, and the expansion of $\log(\tau-e^t)=\log(e^{i\pi})+\log(e^t)-\sum_{k=1}^\infty\frac{e^{-t}\tau^k}{k}$, we find that

$$\begin{aligned}\frac{1}{2\pi i}\int_T \frac{(\alpha_0+\beta_0)\log(\tau-e^t)}{\tau-z}d\tau &= \frac{(\alpha_0+\beta_0)}{2\pi i}\int_T \left\{ i\pi + t - e^{-t}\tau - \frac{e^{-2t}}{2}\tau^2 - ...\right\} \\ &\times \left\{\frac{1}{\tau}+\frac{z}{\tau^2}+\frac{z^2}{\tau^3}+...\right\}d\tau \\ &= (\alpha_0+\beta_0)\left(i\pi + t - e^{-t}z - \frac{e^{-2t}}{2}z^2 - ...\right) \\ &= (\alpha_0+\beta_0)\left(i\pi + t - \sum_{k=1}^{\infty}\frac{e^{-tk}}{k}z^k\right) \\ &= (\alpha_0+\beta_0)\log(z-e^t),\end{aligned} \tag{3.4.31}$$

and for $|z| > 1$,

$$\begin{aligned}\frac{1}{2\pi i}\int_T \frac{(\alpha_0+\beta_0)\log(\tau-e^t)}{\tau-z}d\tau &= \frac{(\alpha_0+\beta_0)}{2\pi i}\int_T \left\{ i\pi + t - e^{-t}\tau - \frac{e^{-2t}}{2}\tau^2 - ...\right\} \\ &\times \left\{\frac{-1}{z}-\frac{\tau}{z^2}-\frac{\tau^2}{z^3}+...\right\}d\tau = 0.\end{aligned} \tag{3.4.32}$$

Similar to the previous integral, but this time by using the expansion

$$\log(\tau-e^{-t}) = \log(\tau) - \sum_{k=1}^{\infty}\frac{e^{-tk}}{k}\tau^{-k}, \tag{3.4.33}$$

we have, for $|z| < 1$,

$$\begin{aligned}\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(\alpha_0-\beta_0)\log(\tau-e^{-t})}{\tau-z}d\tau &= \frac{(\alpha_0-\beta_0)}{2\pi i}\left[\int_{\mathbb{T}}\frac{\log(\tau)}{\tau-z}d\tau + \int_{\mathbb{T}}\Big\{ - e^{-t}\tau^{-1} \right. \\ &\left. - \frac{e^{-2t}}{2}\tau^{-2} - ...\Big\}\left\{\frac{1}{\tau}+\frac{z}{\tau^2}+...\right\}d\tau\right] \\ &= (\alpha_0-\beta_0)\Big(\log(1-z) - i\pi\Big).\end{aligned} \tag{3.4.34}$$

Now, for $|z| > 1$, we continue in the same manner, but now using (3.4.17), we get

$$\begin{aligned} \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(\alpha_0-\beta_0)\log(\tau-e^{-t})}{\tau-z}d\tau &= \frac{(\alpha_0-\beta_0)}{2\pi i}\Bigg[\int_{\mathbb{T}}\frac{\log(\tau)}{\tau-z}d\tau + \int_{\mathbb{T}}\Big\{-e^{-t}\tau^{-1}- \\ &\quad \frac{e^{-2t}}{2}\tau^{-2}-...\Big\}\Big\{-\frac{1}{z}-\frac{\tau}{z^2}-...\Big\}d\tau\Bigg] \\ &= (\alpha_0-\beta_0)\log(z-1)-(\alpha_0-\beta_0)\log z \\ &\quad -(\alpha_0-\beta_0)\Big(-e^{-t}z^{-1}-\frac{e^{-2t}}{2}z^{-2}-...\Big) \\ &= (\alpha_0-\beta_0)\Big(\log(z-1)-\log(z-e^{-t})\Big). \end{aligned} \tag{3.4.35}$$

Utilizing the previous results, for $|z| < 1$, we have

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(-\alpha_0+\beta_0)\log(\tau)}{\tau-z}d\tau = (-\alpha_0+\beta_0)\Big(\log(1-z)-i\pi\Big), \tag{3.4.36}$$

and for $|z| > 1$, we have

$$\frac{1}{2\pi i}\int_{\mathbb{T}} \frac{(-\alpha_0+\beta_0)\log(\tau)}{\tau-z}d\tau = (-\alpha_0+\beta_0)\Big(\log(z-1)-\log z\Big) \tag{3.4.37}$$

and by using (3.4.20), we get

$$\frac{1}{2\pi i}\int_{T} \frac{-i\pi(\alpha_0+\beta_0)}{\tau-z}d\tau = \begin{cases} -i\pi(\alpha_0+\beta_0) & \text{for} \quad |z|<1, \\ 0 & \text{for} \quad |z|>1. \end{cases} \tag{3.4.38}$$

By multiplying all of the above results together, the formula for the Szegő function (3.4.13) is produced. □

Note that the jump matrices $S(z)$ and $N(z)$ are not uniformly close to one another near singularities z_j. Consequently, we must construct local parametrices at these points in the subsequent step. After that, we match them with $N(z)$ on ∂U_{z_j}, and the desired asymptotics is obtained.

3.4.4 Construction of the local parametrices at fixed singularities

Now we will examine separately each intersection where we have opened the lens (Riemann-Hilbert problem for $S(z)$). In the neighbourhood U_{z_j}, we will construct the local parametrices P_{z_j} at $z_j, j = 1, ..., m$. We need a sectionally analytic matrix-valued function that satisfies the following Riemann-Hilbert problem for $P(z_j)$. The study is based on the work of Deift, Its, and Krasovsky [16], [18], and also on [34].

The Riemann-Hilbert problem for $P(z_j)$

RH-P1: $P_{z_j}(z)$ is analytic for $z \in U_{z_j} \setminus \Sigma = \cup_{j=0}^{1}(\Sigma_j \cup \Sigma_j' \cup \Sigma_j'')$.

RH-P2: The boundary values are related by the following jump conditions,

$$P_{z_j,+}(z) = P_{z_j,-}(z) \begin{pmatrix} 1 & 0 \\ f_t(z)^{-1} z^{\pm n} & 1 \end{pmatrix} \quad z \in U_{z_j} \cap (\cup_{j=0}^{1}(\Sigma_j \cup \Sigma_j'')),$$

with the minus sign in the exponent on Σ_j, and plus sign on Σ_j'',

$$P_{z_j,+}(z) = P_{z_j,-}(z) \begin{pmatrix} 0 & f_t(z) \\ -f_t(z)^{-1} & 0, \end{pmatrix} z \in U_{z_j} \cap (\cup_{j=0}^{1}\Sigma_j').$$

RH-P3: For $z \in \partial U_{z_j} \setminus \Sigma$, the following uniform asymptotic holds as $n \to \infty$ (this is known as a matching condition):

$$P_{z_j}(z) N^{-1}(z) = I + o(1). \tag{3.4.39}$$

RH-P4: For $z \to z_j, j = 1, ..., m$, the jump conditions are identical to $S(z)$.

First, we investigate RH-P1, RH-P2, and RH-P4 in order to construct $P_{z_j}(z)$, and then we turn to RH-P3. A series of procedures will be completed to accomplish this construction.

Step 1: Reduction to constant jumps

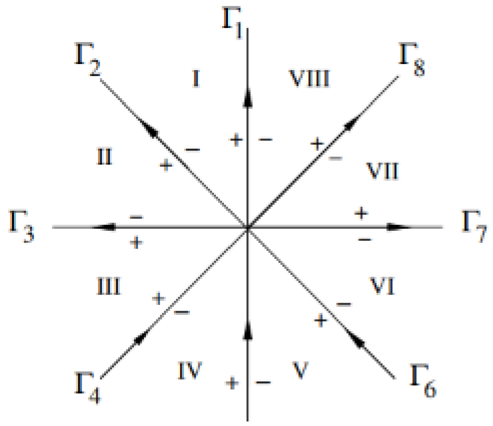

Figure 3.2: Ψ RH problem

First define the following transformation that maps the z-plane to the ξ-plane:

$$\xi(z) = n \log \frac{z}{z_j}, \tag{3.4.40}$$

where $\log z > 0$ when $z > 0$ and the logarithm has a cut on the negative half of the real axis. The neighborhood U_{z_j} is transformed into a neighborhood of zero in the ξ-plane by this transformation. Now, we select the form of the intersection between the neighborhood U_{z_j} and the contour in the Riemann-Hilbert problem for $S(z)$ such that their images are straight lines under this transformation. The function $\xi(z)$ is analytic and one-to-one which maps an arc of the unit circle to an interval along the imaginary axis. The values in sectors I, II, III, IV correspond to the interior of the unit circle, and ξ values in sectors $V, VI, VII, VIII$ correspond to the exterior of the unit circle. In order to address the non-analyticity of $|z - z_j|^{\alpha_j}$, we added an additional jump contour to Σ in the neighborhood of z_j, which is the pre-image of the real lines Γ_3 and Γ_7.

Then, we will present the following auxiliary function $F_j(z)$ defined in [16] by

$$\begin{aligned} F_j(z) = \exp\left\{\frac{1}{2}\log a(z;t)\right\} \prod_{k=1}^{m} (\frac{z}{z_k})^{\beta_k/2} \prod_{k \neq j} h_{\alpha_k}(z) g_{\beta_k}(z)^{1/2} \\ \times h_{\alpha_j}(z) \begin{cases} e^{-i\pi\alpha_j} & \xi \in I, II, V, VI \\ e^{i\pi\alpha_j} & \xi \in III, IV, VII, VIII. \end{cases} \end{aligned} \tag{3.4.41}$$

It is not difficult to show that $F_j(z), j = 1, ..., m$ is analytic in the intersection of each quarter

$\xi-$plane with $\xi(U_{z_j})$ and includes the following jumps:

$$F_{j,+}(z) = F_{j,-}e^{-2\pi i\alpha_j} \quad \xi \in \Gamma_1. \tag{3.4.42}$$

We can show this by using the definition (3.4.41), which states that

$$F_{j,+} = \exp\left\{\frac{1}{2}\log a(z;t)\right\}\prod_{k=1}^{m}(\frac{z}{z_k})^{\beta_k/2}\prod_{k\neq j}h_{\alpha_k}(z)g_{\beta_k}(z)^{1/2}$$
$$\times\, h_{\alpha_j}(z)e^{-i\pi\alpha_j},$$

and

$$F_{j,-} = \exp\left\{\frac{1}{2}\log a(z;t)\right\}\prod_{k=1}^{m}(\frac{z}{z_k})^{\beta_k/2}\prod_{k\neq j}h_{\alpha_k}(z)g_{\beta_k}(z)^{1/2}$$
$$\times\, h_{\alpha_j}(z)e^{i\pi\alpha_j}.$$

Thus, we derived the jump at $\xi \in \Gamma_1$ by comparing $F_{j,+}$ and $F_{j,-}$. Similarly, we can confirm the jumps in other sectors.

$$F_{j,+}(z) = F_{j,-}e^{2\pi i\alpha_j} \quad \xi \in \Gamma_5. \tag{3.4.43}$$

$$F_{j,+}(z) = F_{j,-}e^{\pi i\alpha_j} \quad \xi \in \Gamma_3 \cup \Gamma_7. \tag{3.4.44}$$

Using the function (3.4.19) of $f_t(z)$ and the definition of the function $F_j(z)$ in (3.4.41), the following relations are obtained,

$$F_j(z)^2 = f_t(z)e^{-2i\pi\alpha_j}g_{\beta_j}^{-1}(z), \qquad \xi \in I, II, V, VI \tag{3.4.45}$$

$$F_j(z)^2 = f_t(z)e^{2i\pi\alpha_j}g_{\beta_j}^{-1}(z), \qquad \xi \in III, IV, VII, VIII \tag{3.4.46}$$

Then, we search for a function $P_{z_j}(z)$ of the following form:

$$P_{z_j}(z) = P^{(1)} \begin{pmatrix} F_j^{-1} & 0 \\ 0 & F_j \end{pmatrix} z^{\pm n\sigma_3/2}.$$

The plus sign is used when $|z| < 1$ and minus sign when $|z| > 1$, which corresponds to $\xi \in I, II, III, IV$ and $\xi \in V, VI, VII, VIII$, respectively. In order to satisfy the conditions from (RH-P1),(RH-P2), and (RH-P4), $P^{(1)}(z)$ must be constant.

Step 2: Riemann-Hilbert problem for $\Psi_j(\xi)$ and the solution

Using the map in (3.4.40), this constant jump problem $P^{(1)}$ produces a model problem $\Psi_j(\xi)$ which is defined in ξ-plane. Set

$$P^{(1)}(z) = \Psi_j(\xi), \tag{3.4.47}$$

where $\Psi_j(\xi)$ satisfies the Riemann-Hilbert problem on the contour $\Gamma = \cup_{j=1}^{8} \Gamma_j$,

RH-$\Psi_j 1$: Ψ_j is analytic for $\xi \in \mathbb{C} \setminus \Gamma$.

RH-$\Psi_j 2$: Ψ_j satisfies the following jump conditions:

$$\Psi_{j,+}(\xi) = \Psi_{j,-}(\xi) \begin{pmatrix} 0 & e^{-i\pi\beta_j} \\ -e^{-i\pi\beta_j} & 0 \end{pmatrix}, \qquad \xi \in \Gamma_1, \tag{3.4.48}$$

$$\Psi_{j,+}(\xi) = \Psi_{j,-}(\xi) \begin{pmatrix} 0 & e^{-i\pi\beta_j} \\ -e^{i\pi\beta_j} & 0 \end{pmatrix}, \qquad \xi \in \Gamma_5, \tag{3.4.49}$$

$$\Psi_{j,+}(\xi) = \Psi_{j,-}(\xi) e^{i\pi\alpha_j\sigma_3}, \qquad \xi \in \Gamma_3 \cup \Gamma_7, \tag{3.4.50}$$

$$\Psi_{j,+}(\xi) = \Psi_{j,-}(\xi) \begin{pmatrix} 1 & 0 \\ -e^{\pm i\pi(\beta_j - 2\alpha_j)} & 1 \end{pmatrix}, \tag{3.4.51}$$

where the plus sign in the exponent for $\xi \in \Gamma_2$, and the minus sign for $\xi \in \Gamma_4$

$$\Psi_{j,+}(\xi) = \Psi_{j,-}(\xi) \begin{pmatrix} 1 & 0 \\ -e^{\pm i\pi(\beta_j + 2\alpha_j)} & 1 \end{pmatrix}, \tag{3.4.52}$$

with plus sign for $\xi \in \Gamma_8$, and the minus sign for $\xi \in \Gamma_6$.

RH-$\Psi_j 3$: As $\xi \to 0$ with $\xi \in \mathbb{C} \setminus \Gamma$ outside the lenses,

$$\Psi_j(z) = \begin{pmatrix} \mathcal{O}(1) & \mathcal{O}(\log|\xi|) \\ \mathcal{O}(1) & \mathcal{O}(\log|\xi|) \end{pmatrix}, \quad \alpha_j = 0,\ \beta_j \neq 0, \tag{3.4.53}$$

and

$$\Psi_j(z) = \begin{pmatrix} \mathcal{O}(\xi^{\alpha_j}) & \mathcal{O}(\xi^{\alpha_j}) + \mathcal{O}(\xi^{-\alpha_j}) \\ \mathcal{O}(\xi^{\alpha_j}) & \mathcal{O}(\xi^{\alpha_j}) + \mathcal{O}(\xi^{-\alpha_j}) \end{pmatrix}, \quad \alpha_j \neq 0. \tag{3.4.54}$$

The solution of this problem was explicitly solved in terms of confluent hypergeometric functions (see, [16]).

Proposition 3.4.2. *([16], Proposition 4.1) Let $\alpha_j \pm \beta_j \notin \mathbb{Z}_-$ for all $j = 1, ..., m$. Then a solution for the $\Psi_j(\xi)$ Riemann-Hilbert problem in sector I is given by:*

$$\Psi_j(\xi) = \Psi_j^{(I)}(\xi) = \begin{pmatrix} \Psi_{j,11} & \Psi_{j,12} \\ \Psi_{j,21} & \Psi_{j,22} \end{pmatrix}, \tag{3.4.55}$$

where

$$\Psi_{j,11} = \xi^{\alpha_j} \psi(\alpha_j + \beta_j, 1 + 2\alpha_j, \xi) e^{i\pi(2\beta_j + \alpha_j)} e^{-\xi/2}$$

$$\Psi_{j,12} = -\xi^{\alpha_j} \psi(1 + \alpha_j - \beta_j, 1 + 2\alpha_j, e^{-i\pi}\xi) e^{i\pi(\beta_j + \alpha_j)} e^{\xi/2} \frac{\Gamma(1 + \alpha_j - \beta_j)}{\Gamma(\alpha_j + \beta_j)}$$

$$\Psi_{j,21} = -\xi^{-\alpha_j} \psi(1 - \alpha_j + \beta_j, 1 - 2\alpha_j, \xi) e^{i\pi(\beta_j - 3\alpha_j)} e^{-\xi/2} \frac{\Gamma(1 + \alpha_j + \beta_j)}{\Gamma(\alpha_j - \beta_j)}$$

and

$$\Psi_{j,22} = \xi^{-\alpha_j}\psi(-\alpha_j - \beta_j, 1 - 2\alpha_j, e^{-i\pi\xi})e^{-i\pi\alpha_j}e^{\xi/2}.$$

The solutions for the other sectors are reconstructed by using (3.4.48)-(3.4.52). **Step 3: The matching condition**

We multiply $\Psi_j(\xi)$ by the analytic function $E(z)$ from the left. Then we get

$$P^{(1)}(z) = E(z)\Psi_j(\xi). \tag{3.4.56}$$

The precise $E(z)$ will be given later. Thus the parametrices that we have are

$$P_{z_j} = E(z)\Psi_j(\xi)F_j^{-\sigma_3/2}z^{\pm n\sigma_3/2}. \tag{3.4.57}$$

Then, on the boundary, we will match this solution with $N(z)$ for large n. To accomplish this, $E(z)$ must be close to the following

$$N(z)\begin{pmatrix} F_j(z) & 0 \\ 0 & F_j^{-1}(z) \end{pmatrix} z^{\mp n\sigma_3/2}\Psi_j^{-1}(\xi).$$

The limit for $z \in \partial U_{z_j}$ and $n \to \infty$ is equivalent to $\xi \to \infty$. Consequently, we must determine the asymptotic expansion of $\Psi_j(\xi)$ using the classical results in [8] and [28] for the confluent hypergeometric function $\Psi_j(\xi)$. As $|x| \to \infty$ with $-\frac{3\pi}{2} < \arg x < \frac{3\pi}{2}$,

$$\psi(a, c, x) = x^{-a}[1 - a(1 + a - c)x^{-1} + \mathcal{O}(x^{-2})]. \tag{3.4.58}$$

In addition, this result can be applied to $\psi(a, c, \xi)$ and $\psi(a, c, e^{-i\pi}\xi)$ for $0 < \arg\xi < 2\pi$ when $\xi \in I$.

Applying this result to (3.4.55), we obtain the asymptotics as $\xi \to \infty$ in sector I, which remains unchanged in sector II because of the correct triangular structure of the jump matrices.

$$\begin{aligned}\Psi_j^{(I)} &= \Psi_j^{(II)}(\xi)\\ &= \left[I + \frac{1}{\xi}\begin{pmatrix} \alpha_j^2 - \beta_j^2 & \frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}e^{i\pi(\beta_j+4\alpha_j)} \\ -\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}e^{-i\pi(\beta_j+4\alpha_j)} & -(\alpha_j^2-\beta_j^2)\end{pmatrix} + \mathcal{O}(\xi^{-2})\right]\\ &\times \xi^{-\beta_j\sigma_3}e^{-\xi\sigma_3/2}\begin{pmatrix} e^{i\pi(2\beta_j+\alpha_j)} & 0 \\ 0 & e^{-i\pi(\beta_j+2\alpha_j)}\end{pmatrix},\\ &\xi\to\infty, \xi\in I, II, \alpha_j\pm\beta_j\neq -1,-2,...\end{aligned} \tag{3.4.59}$$

To achieve the asymptotic behaviuor in other sectors, we employ the relevant jump matrices that yield the following results:

$$\Psi_j^{(III)}(\xi) = \Psi_j^{(IV)}(\xi) = \Psi_j^{(I)}(\xi)e^{i\pi\alpha_j\sigma_3}. \tag{3.4.60}$$

$$\Psi_j^{(V)}(\xi) = \Psi_j^{(VI)}(\xi) = \Psi_j^{(I)}(\xi)\begin{pmatrix} 0 & -e^{i\pi\beta_j} \\ e^{-i\pi\beta_j} & 0\end{pmatrix}e^{-i\pi\alpha_j\sigma_3}. \tag{3.4.61}$$

$$\Psi_j^{(VIII)}(\xi) = \Psi_j^{(VII)}(\xi) = \Psi_j^{(I)}(\xi)\begin{pmatrix} 0 & -e^{-i\pi\beta_j} \\ e^{i\pi\beta_j} & 0\end{pmatrix}. \tag{3.4.62}$$

The following result is obtained by substituting this asymptotics into the condition on $E(z)$ (see (3.4.39)):

$$P_{z_j}(z)N^{-1}(z) = E(z)\Psi_j(\xi)F_j(z)^{-\sigma_3}z^{\pm\frac{n\sigma}{2}}N^{-1} = I + o(1). \tag{3.4.63}$$

The function $E(z)$ is then obtained in sectors I and II as follows:

$$E(z) = N(z)\xi^{\beta_j\sigma_3}F_j^{-\sigma_3}z_j^{-\frac{n\sigma_3}{2}}\begin{pmatrix} e^{-i\pi(2\beta_j+\alpha_j)} & 0 \\ 0 & e^{i\pi(\beta_j+2\alpha_j)}\end{pmatrix}, \quad \xi\in I, II \tag{3.4.64}$$

For more information on how to find the asymptotics of the matrices $E(z)$ for the remaining sectors using the asymptotics in (3.4.59), see [16] and in particular (4.42)-(4.50) therein.

Thus by applying (3.4.47), (3.4.57), (3.4.64), and Proposition 3.4.2, the construction of the parametrices P_{z_j} has been completed for U_{z_j}.

Similarly to [16], we derive the unique expansions in $u = z - z_j$ as $u \to 0$. Recall (3.1.1), and the factorisation of $V(z)$, and from (3.4.13), (3.4.19),(3.4.40), (3.4.41), we get

$$F_j(z) = \eta_j e^{-3i\pi\alpha/2} z_j^{-\alpha_j} u^{\alpha_j}(1 + \mathcal{O}(u)), \quad u = z - z_j \qquad \xi \in I, \tag{3.4.65}$$

where

$$\eta_j = \exp\{\log a(z_j;t)/2\} \exp\left\{ -\frac{i\pi}{2}\Big(\sum_{k=1}^{j-1}\beta_k - \sum_{k=j+1}^{m}\beta_k\Big)\right\} \prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\beta_k/2} |z_j - z_k|^{\alpha_k}, \tag{3.4.66}$$

$$\begin{aligned} D(z) = &\prod_{j=1}^{m} u^{\alpha_j+\beta_j} z_j^{-(\alpha_j+\beta_j)} e^{-i\pi(\alpha_j+\beta_j)} (z - e^t)^{\alpha_0+\beta_0} e^{-i\pi(\alpha_0+\beta_0)} \\ &\exp\left\{\sum_{k=0}^{\infty} V_k z_j^k\right\}(1 + \mathcal{O}(u)), \end{aligned} \tag{3.4.67}$$

and

$$\xi(z) = n \log\frac{z}{z_j} + \mathcal{O}(u^2) = n\frac{u}{z_j}(1 + \mathcal{O}(u)). \tag{3.4.68}$$

By combining them, we obtain the following:

$$\begin{aligned} \left(\frac{D(z)}{\xi^{\beta_j} F_j(z)}\right)^2 = &\, e^{V_0} \frac{\exp\sum_{k=1}^{\infty} V_k z_j^k}{\exp\sum_{k=1}^{\infty} V_{-k} z_j^{-k}} (1 - z_j e^{-t})^{(\alpha_0+\beta_0)} (1 - e^{-t} z_j^{-1})^{-(\alpha_0-\beta_0)} \\ &\times n^{-2\beta_j} e^{t(\alpha_0+\beta_0)} \times e^{i\pi(\alpha_j - 2\beta_j)} \times \prod_{k\neq j}\left(\frac{z_j}{z_k e^{i\pi}}\right)^{\alpha_k} |z_j - z_k|^{2\beta_k}(1 + \mathcal{O}(u)). \end{aligned} \tag{3.4.69}$$

According to (3.4.64), $\det E(z) = e^{i\pi(\alpha_j - \beta_j)}$ holds in all ξ-plane sectors. It is also essential to remember that $\det \psi_j(\xi) = e^{-i\pi(\alpha_j-\beta_j)}$. According to Liouville's theorem, $\det \psi_j(\xi)$ contains no jumps. As a result, the singularities at zero are removable as $\Re\alpha_j > -\frac{1}{2}$, and the constant value of the function can be determined by applying (3.4.59). By combining these observations and noting that $\det P_{z_j} = 1$, it is possible to conclude that P_{z_j} is unique and that its inverse exists, and so, $P_{z_j}(z)\hat{P}_{z_j}(z) = I$. It is also important to note that by constructing $P_{z_j}(z)$, which has the same jumps as $S(z)$, it follows that $S(z)P_{z_j}(z)^{-1}$ has no jumps, so it is analytic near z_j. This means that the singularities are removable. The singularities at $z = z_j$ are at most $\mathcal{O}(|z-z_j|^{2\alpha_j})$ or $\mathcal{O}(\log|z-z_j|)$, which can be seen by applying (3.4.8), (3.4.9), (3.4.53), and (3.4.54).

Observe that the error term in the equation (3.4.63) is

$$o(1) = n^{-\Re\beta_j\sigma_3}\mathcal{O}(\frac{1}{n})n^{\Re\beta_j\sigma_3} \quad \text{if} \quad \Re\beta_j \in (-\frac{1}{2}, \frac{1}{2}).$$

Next, we will extend (3.4.63) to figure out the first correction term $\Delta_1(z)$ in the asymptotic series in inverse powers of n satisfying

$$P_{z_j}(z)N^{-1}(z) = I + \Delta_1(z) + n^{-\Re\beta_j\sigma_3}\mathcal{O}(n^{-2})n^{\Re\beta_j\sigma_3}. \tag{3.4.70}$$

By considering further terms, we can then extend (3.4.59) to discover a complete asymptotic series. By multiplying out the matrices in (3.4.64), we obtain

$$E(z) = \begin{pmatrix} 0 & D(z)\xi^{-\beta_j}F_j^{-1}z_j^{n/2}e^{i\pi(\beta_j+2\alpha_j)} \\ -E_{12}^{-1}e^{i\pi(\alpha_j-\beta_j)} & 0 \end{pmatrix}. \tag{3.4.71}$$

In addition,

$$P_{z_j}N^{-1} = \begin{pmatrix} E_{12}\Psi_{1,22}F_jz^{-n/2}D(z)^{-1} & -E_{12}\Psi_{1,21}F_j^{-1}z^{n/2}D(z) \\ E_{21}\Psi_{1,12}F_jz^{-n/2}D(z)^{-1} & -E_{21}\Psi_{1,11}F_j^{-1}z^{n/2}D(z) \end{pmatrix}. \tag{3.4.72}$$

After that , by substituting in (3.4.71), and using (3.4.59) for $\psi_{j,im}$ yields the following result:

$$\Delta_1(z) = \frac{1}{\xi}\begin{pmatrix} -(\alpha_j^2-\beta_j^2) & z_j^n\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}\left(\frac{D(z)}{\xi^{\beta_j}F_j}\right)^2e^{i\pi(2\beta_j-\alpha_j)} \\ -z_j^n\frac{\Gamma(1+\alpha_j-\beta_j)}{\Gamma(\alpha_j+\beta_j)}\left(\frac{D(z)}{\xi^{\beta_j}F_j}\right)^{-2}e^{-i\pi(2\beta_j-\alpha_j)} & (\alpha_j^2-\beta_j^2) \end{pmatrix}. \tag{3.4.73}$$

Then, we will look for the 12 element of the correction that will be required in what follows:

$$
\begin{aligned}
(\Delta_1(z))_{12} = \frac{1}{\xi}\sum_{j=1}^{m}\Bigg[& z_j^n e^{V_0}\exp\Big\{\sum_{k=1}^{\infty}V_k z_j^k\Big\}\exp\Big\{-\sum_{k=1}^{\infty}V_{-k}z_j^{-k}\Big\}(1-z_je^{-t})^{(\alpha_0+\beta_0)} \\
& \times(1-e^{-t}z_j^{-1})^{-(\alpha_0-\beta_0)}\times e^{t(\alpha_0+\beta_0)}\times\prod_{k\neq j}\left(\frac{z_j}{z_k}\right)^{\alpha_k}|z_j-z_k|^{2\beta_k} \\
& \times\exp\Big\{-i\pi\Big(\sum_{k=1}^{j-1}\alpha_k-\sum_{k=j+1}^{m}\alpha_k\Big)\Big\}n^{-2\beta_j}\frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}\Bigg](1+\mathcal{O}(u)).
\end{aligned}
\tag{3.4.74}
$$

In addition, we observe,

$$
\frac{1}{\xi}=\frac{z_j}{n(z-z_j)}+\mathcal{O}(z-z_j)^2 \quad z\to z_j \tag{3.4.75}
$$

3.4.5 Local parametrix at emerging singularity

Before constructing the local parametrix at $z_0 = 1$, we will present the Riemann-Hilbert problem for Painlevé V equation, which plays an important role in describing the transition in [13] and [34]. For the reader's convenience, we recall the relevant theory.

Riemann Hilbert-Problem for Painlevé V

We consider the second order ordinary differential equation

$$
\left(x\frac{d^2\sigma}{dx^2}\right)^2=\left(\sigma-x\frac{d\sigma}{dx}+2(\frac{d\sigma}{dx})^2+2\alpha_0\frac{d\sigma}{dx}\right)^2-4\left(\frac{d\sigma}{dx}\right)^2\left(\frac{d\sigma}{dx}+\alpha_0+\beta_0\right)\left(\frac{d\sigma}{dx}+\alpha_0-\beta_0\right), \tag{3.4.76}
$$

which is the σ-form of the painlevé V equation

$$
u_{xx}=\left(\frac{1}{2u}+\frac{1}{u-1}\right)u_x^2-\frac{1}{x}u_x+\frac{(u-1)^2}{x^2}\left(Au+\frac{b}{u}\right)+\frac{Cu}{x}+D\frac{u(u+1)}{u-1} \tag{3.4.77}
$$

studied by Jimbo, Miwa, and Okamoto in [30] (see also [31]).

For the equation above, we have the following parameters,

$$
A=\frac{1}{2}(\alpha_0-\beta_0)^2,\quad B=-\frac{1}{2}(\alpha_0+\beta_0)^2,\quad C=1+2\beta_0,\quad D=-\frac{1}{2}. \tag{3.4.78}
$$

According to [13], the Painlevé V equation has a solution that can be written as follows:

$$\sigma(x) = \int_x^{+\infty} v(\xi) d\xi. \tag{2.4.79}$$

Furthermore, the function $\sigma(x)$ can be formed directly in terms of the Riemann-Hilbert problem given below. Consider the contour $\Gamma = \bigcup_{j=1}^{6} \Gamma_j$ in the complex plane $\mathbb{C}$ with

$$\begin{aligned} &\Gamma_1 = \frac{1}{2} + e^{i\pi/4}\mathbb{R}^+, \quad \Gamma_2 = \frac{1}{2} + e^{i3\pi/4}\mathbb{R}^+, \quad \Gamma_3 = \frac{1}{2} + e^{i5\pi/4}\mathbb{R}^+, \\ &\Gamma_4 = \frac{1}{2} + e^{i7\pi/4}\mathbb{R}^+, \quad \Gamma_5 = (1, \infty), \quad \Gamma_6 = (0, 1). \end{aligned} \tag{3.4.80}$$

The Riemann-Hilbert problem for Ψ.

(RH-$\Psi 1$): Ψ is analytic for $\xi \in \mathbb{C} \setminus \Gamma$.

(RH-$\Psi 2$): Ψ has a continuous boundary value on $\Gamma \setminus \{0, \frac{1}{2}, 1\}$ with the jump conditions given as follows:

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 1 & e^{i\pi(\alpha_0 - \beta_j)} \\ 0 & 1 \end{pmatrix}, \quad \xi \in \Gamma_1, \tag{3.4.81}$$

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 1 & 0 \\ -e^{-i\pi(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \xi \in \Gamma_2, \tag{3.4.82}$$

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 1 & 0 \\ e^{i\pi(\alpha_0 - \beta_0)} & 1 \end{pmatrix}, \quad \xi \in \Gamma_3, \tag{3.4.83}$$

$$\Psi_+(\xi) = \Psi_-(\xi) \begin{pmatrix} 1 & -e^{-i\pi(\alpha_0 - \beta_0)} \\ 0 & 1 \end{pmatrix} \quad \xi \in \Gamma_4. \tag{3.4.84}$$

$$\Psi_+(\xi) = \Psi_-(\xi) e^{2\pi i \beta_0 \sigma_3} \quad \xi \in \Gamma_5, \tag{3.4.85}$$

$$\Psi_+(\xi) = \Psi_-(\xi) e^{-\pi i(\alpha_0 - \beta_0)\sigma_3} \quad \xi \in \Gamma_6. \tag{3.4.86}$$

(RH-$\Psi 3$): As $\xi \to \infty$, Ψ has the following asymptotic behaviour for certain matrices

$C_1 = C_1(x, \alpha, \beta)$,

$$\Psi(\xi) = \left(I + \frac{C_1}{\xi} + \mathcal{O}(\xi^{-2})\right)\xi^{-\beta_0\sigma_3} e^{-(\frac{x}{2})\xi\sigma_3}. \tag{3.4.87}$$

(RH-Ψ4): Ψ has the following limiting behaviour

(a)

$$\Psi(\xi) = \mathcal{O}\begin{pmatrix} |\xi|^{(\alpha_0-\beta_0)/2} & |\xi|^{-(\alpha_0-\beta_0)/2} \\ |\xi|^{(\alpha_0-\beta_0)/2} & |\xi|^{-(\alpha_0-\beta_0)/2} \end{pmatrix} \quad \text{as} \quad \xi \to 0. \tag{3.4.88}$$

(b)

$$\Psi(\xi) = \mathcal{O}\begin{pmatrix} |\xi-1|^{-(\alpha_0+\beta_0)/2} & |\xi-1|^{(\alpha_0+\beta_0)/2} \\ |\xi-1|^{-(\alpha_0+\beta_0)/2} & |\xi-1|^{(\alpha_0+\beta_0)/2} \end{pmatrix} \quad \text{as} \quad \xi \to 1. \tag{3.4.89}$$

(c) Ψ is bounded in a neighborhood of $1/2$.

Now, we will construct the parametrix at $z_0 = 1$. Similar to the parametrices at $z_j, j = 1, ..., m$, P_{z_0} will be constructed at $z_0 = 1$ which satisfies the same jump conditions as the Riemann-Hilbert problem for $S(z)$ in the neighborhood of z_0 with a small fixed radius. Then, on the boundary of U_{z_0}, we will have a matching condition with the Riemann-Hilbert problem for $N(z)$. This will be accomplished by following a series of steps similarly to [13] and [34].

Step 1: Assuming the Riemann-Hilbert problem for Ψ is solvable, define the following function,

$$\Phi(\lambda; x) = e^{\frac{x}{4}\sigma_3} x^{-\beta_0\sigma_3} \Psi\left(\frac{\lambda}{x} + \frac{1}{2}; x\right) G(\lambda; x)^{\frac{1}{2}\sigma_3} e^{\pm\frac{\pi i}{2}(\alpha_0-\beta_0)\sigma_3} \tag{3.4.90}$$

for $\pm\Im\lambda > 0$, respectively. The function $G(\lambda; x)$ is analytic in $\mathbb{C} \setminus ((-\infty, -\frac{x}{2}] \cup [\frac{x}{2}, +\infty))$ and defined as follows:

$$G(\lambda; x) = (\lambda + \frac{x}{2})^{-(\alpha_0-\beta_0)}(\lambda - \frac{x}{2})^{(\alpha_0+\beta_0)} e^{\lambda} e^{-\pi i(\alpha_0-\beta_0)}, \quad x > 0, \tag{3.4.91}$$

where $-\pi < \arg(\lambda + \frac{x}{2}) < \pi$, $0 < \arg(\lambda - \frac{x}{2}) < 2\pi$.

The matrix-valued function $\Phi(\lambda; x)$ in (3.4.90) solves the following Riemann-Hilbert problem for $x > 0$:

(RH-$\Phi1$): $\Phi : \mathbb{C} \setminus \cup_{j=1}^{4} e^{\pi i(2j-1)/4}\mathbb{R}^+ \to \mathbb{C}^{2\times 2}$ is analytic, with the rays $e^{\pi i(2j-1)/4}\mathbb{R}^+$ as shown in Figure 3.3.

(RH-($\Phi2$): Φ has the following jump conditions on $\cup_{j=1}^{4} e^{\pi i(2j-1)/4}\mathbb{R}^+ \setminus \{0\}$:

$$\Phi_+(\lambda) = \Phi_-(\lambda) \begin{pmatrix} 1 & G(\lambda;x)^{-1} \\ 0 & 1 \end{pmatrix}, \quad \text{as} \quad \lambda \in (e^{\pi i/4}\mathbb{R}^+ \cup e^{7\pi i/4}\mathbb{R}^+) \tag{3.4.92}$$

and

$$\Phi_+(\lambda) = \Phi_-(\lambda) \begin{pmatrix} 1 & 0 \\ -G(\lambda;x) & 1 \end{pmatrix}, \quad \text{as} \quad \lambda \in (e^{3\pi i/4}\mathbb{R}^+ \cup e^{5\pi i/4}\mathbb{R}^+). \tag{3.4.93}$$

(RH-$\Phi3$): Φ has the following behaviour as $\lambda \to \infty$,

$$\Phi(\lambda) = I + \mathcal{O}(\lambda^{-1}). \tag{3.4.94}$$

(RH-$\Phi4$): Φ is bounded near 0.

Proposition 3.4.3 (Proposition 3.1 of [13]). *We have*

1. *If $\Re\alpha_0 > -1/2$, the Riemann-Hilbert problem for Φ has a unique solution for all possible finite numbers of positive $x-$values $\{x_1, ..., x_k\}$, where $x_j = x_j(\alpha, \beta)$ and $k = k(\alpha, \beta)$*

2. *If $\alpha_0 > -1/2$, (α_0 is real), and $\Re\beta_0 = 0$, then the Riemann-Hilbert problem for Φ has a unique solution for all $x > 0$.*

3. *If $\Re\alpha_0 > -1/2$, the asymptotics for Φ as $\lambda \to \infty$ is valid uniformly for $x \in (0, \infty)$, with the exception of the set $\{x_1, ..., x_k\}$.*

The preceding proposition was proved in [13] by Clyaeys, Its, and Krasovsky.

Step 2: We will map the jump matrices for Φ into the jump matrices for S in the neighbourhood of $z_0 = 1$ by defining the conformal mapping that maps 1 to 0 and $e^{\pm t}$ to $\pm x/2$, respectively.

$$\lambda(z) = \frac{x}{2t}\log(z), \tag{3.4.95}$$

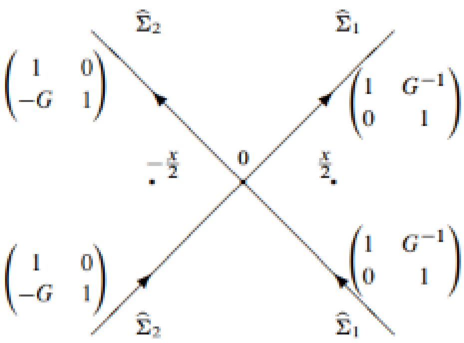

Figure 3.3: Φ RH-problem

where $\log z > 0$ when $z > 0$ and the branch is cut along the negative axis. Consequently, we require that $e^{\lambda}(z) = z^n$, and so,

$$x = 2nt. \tag{3.4.96}$$

Here $\lambda(z)$ translates Σ_1 and Σ_2 close to 1 onto the jump contour $\cup_{j=1}^{4} e^{i\pi(2j-1)/4\mathbb{R}+}$ for Φ.

Step 3: We search for the parametrix P_{z_0} with the form:

$$P_{z_0}(z) = E(z)\Phi(\lambda(z); 2nt)W(z), \tag{3.4.97}$$

where $W(z)$ is given by,

$$W(z) = \begin{cases} -G(\lambda(z))^{-\sigma_3/2} z^{n\sigma_3/2} f_t(z)^{-\sigma_3/2}\sigma_3, & \text{for} \quad |z| < 1 \\ G(\lambda(z))^{-\sigma_3/2} z^{n\sigma_3/2} f_t(z)^{\sigma_3/2}\sigma_1, & \text{for} \quad |z| > 1 \end{cases} \tag{3.4.98}$$

and

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Noting that the branch points of G cancel with those of f_t, so $W(z)$ is analytic in $U_{z_0} \setminus \mathbb{T}$. To satisfy the conditions for the construction in which P_{z_0} has the same jump condition as $S(z)$, we must assume that $E(z)$ is analytic in the neighborhood of z_0.

Step 4: We attempt to fix $E(z)$ in a way that satisfies the matching condition $P_{z_0}N^{-1} = I + o(1)$ on the boundary of U_{z_0}. Consider the behaviour of the parametrix P_{z_0} on the boundary of U_{z_0}. By utilising (3.4.95), we have

$$|\lambda(z)| > cn \quad z \in \partial U_{z_0}$$

Using Proposition 3.4.3, as $n \to \infty$ and $2nt$ remains bounded away from $\{x_1, x_2, ..., x_k\}$, we can use the asymptotic behaviour of $\Phi(z)$ in (3.4.94) to demonstrate that $\Phi(z)$ approaches identity as $z \to \infty$, and we have

$$P_{z_0}(z) = E(z)(I + \mathcal{O}(\frac{1}{n}))W(z), \quad \text{as} \quad n \to \infty \tag{3.4.99}$$

uniformly with respect to $0 < t < t_0$ and $z \in \partial U_{z_0}$. We assume that $e^{\pm t}$ lie inside U_{z_0} for small t_0. As $n \to \infty$, by (3.4.91) and (3.4.98), we have

$$W(z) = n^{-\beta_0} \begin{cases} \begin{pmatrix} \mathcal{O}(1) & 0 \\ 0 & \mathcal{O}(1) \end{pmatrix}, & |z| < 1 \\ \begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix}, & |z| > 1 \end{cases} \tag{3.4.100}$$

uniformly in terms of $0 < t < t_0$ and $z \in \partial U_{z_0} \setminus \mathbb{T}$. Then we set

$$E(z) = N(z)W(z)^{-1}. \tag{3.4.101}$$

By utilizing the jump for $N(z)$ and $W(z)$, we can easily show that $E(z)$ is analytic in the whole area $\overline{U}_{z_0}$ of 1. Then, from (3.4.11) and (3.4.101), we get the following asymptotics:

$$E(z) = \begin{pmatrix} 0 & \mathcal{O}(1) \\ \mathcal{O}(1) & 0 \end{pmatrix} n^{\beta_0 \sigma_3} \quad \text{as} \quad n \to \infty \tag{3.4.102}$$

uniformly for $0 < t < t_0$, and $z \in \partial U_{z_0}$. The asymptotics for the matching condition for

$z \in \partial U_{z_0}$ can be obtained using this and (3.4.99):

$$\begin{aligned} P_{z_0}(z)N(z)^{-1} &= E(z)(I+\mathcal{O}(1/n))E(z)^{-1} \\ &= I + n^{-\beta_0\sigma_3}\mathcal{O}(1/n)n^{\beta_0\sigma_3} \text{as} \quad n \to \infty \end{aligned} \tag{3.4.103}$$

for $0 < t < t_0$ uniformly. Thus, we conclude the construction of the local parametrix at z_0 holds if $2nt$ is away from the set $\{x_1, ..., x_k\}$.

The singularity at z_0 is removable because P_{z_0} has the same jump conditions as $S(z)$, so $S(z)P_{z_0}(z)^{-1}$ is analytic in a neighborhood of $z_0 = 1$.

Now we will compute the first correction term $\Delta_1(z)$ in the asymptotic series in inverse powers of n:

$$P_{z_0}(z)N(z)^{-1} = I + \Delta_1(z) + n^{-\Re\beta_0\sigma_3}\mathcal{O}(1/n^2)n^{\Re\beta_0\sigma_3}. \tag{3.4.104}$$

Then, utilising (3.4.98), (3.4.101) and (3.4.11), we obtain

$$P_{z_0}(z)N(z)^{-1} = \begin{cases} D(z)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} W(z)^{-1}\Phi(\lambda(z))W(z)\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} D(z)^{-\sigma_3}, & |z| < 1 \\ D(z)^{\sigma_3}W(z)^{-1}\Phi(\lambda(z))W(z)D(z)^{-\sigma_3}, & |z| > 1. \end{cases} \tag{3.4.105}$$

Denoting the elements of the matrix $\Phi(\lambda(z))$ by $= \Phi_{ij}$ for $i, j = 1, 2$, setting $G(\lambda(z), x) = G$, and applying (3.4.98), we get

$$P_{z_0}(z)N(z)^{-1} = \begin{cases} \begin{pmatrix} \Phi_{22} & D(z)^2G^{-1}z^n f_t(z)^{-1}\phi_{21} \\ D(z)^{-2}Gz^{-n}f_t(z)\Phi_{12} & \Phi_{11} \end{pmatrix} & \text{for} |z| < 1 \\ \begin{pmatrix} \Phi_{22} & D(z)^2G^{-1}z^n f_t(z)\phi_{21} \\ D(z)^{-2}Gz^{-n}f_t(z)\Phi_{12} & \Phi_{11} \end{pmatrix} & \text{for } |z| > 1. \end{cases} \tag{3.4.106}$$

To figure out the asymptotic behaviour of these, the behaviour is evaluated as $z \to e^{\pm t}$. Choosing the convenient region $|z| \lesseqgtr 1$ and taking $u = z - e^{\pm t} \to 0$, by the Szegő function defined in (3.4.13), we obtain,

$$D(z)=\begin{cases}\prod_{j=1}^m(1-z_j^{-1}e^{-t})^{\alpha_j+\beta_j}(1-e^{-2t})^{\alpha_0+\beta_0}e^{t(\alpha_0+\beta_0)}\exp\Big\{\sum_{k=0}^\infty V_ke^{-tk}\Big\}(1+\mathcal{O}(u)) & \text{as } z\to e^{-t}\\ \prod_{j=1}^m(1-z_je^{-t})^{-\alpha_j+\beta_j}(1-e^{-2t})^{-\alpha_0+\beta_0}\exp\Big\{-\sum_1^\infty V_{-k}e^{-tk}\Big\}(1+\mathcal{O}(u) & \text{as } z\to e^t.\end{cases} \tag{3.4.107}$$

By (3.4.91),

$$\begin{aligned}G(\lambda(z);x)&=n^{2\beta_0}(\log\frac{z}{e^{-t}})^{-(\alpha_0-\beta_0)}(\log\frac{z}{e^t})^{\alpha_0-\beta_0}z^ne^{-i\pi(\alpha_0-\beta_0)}\\ &=\begin{cases}n^{2\beta_0}u^{-(\alpha_0-\beta_0)}e^{-t(\alpha_0-\beta_0)}(2t)^{\alpha_0+\beta_0}e^{-tn}e^{2i\pi\beta_0}(1+\mathcal{O}(u)) & z\to e^{-t}\\ n^{2\beta_0}u^{\alpha_0+\beta_0}e^{-t(\alpha_0+\beta_0)}(2t)^{-\alpha_0+\beta_0}e^{tn}e^{-i\pi(\alpha_0-\beta_0)}(1+\mathcal{O}(u)) & z\to e^t.\end{cases}\end{aligned} \tag{3.4.108}$$

Taking into account the symbols (3.1.1) and (3.4.19), we get

$$f(z;t)=\begin{cases}e^{V(e^{-t})}\prod_{j=1}^m(1-z_j^{-1}e^{-t})^{2\alpha_j}e^{t(\alpha_j-\beta_j)}e^{i\pi(\alpha_j-\beta_j)}z_j^{(\alpha_j-\beta_j)} & \\ \times(1-e^{-2t})^{\alpha_0+\beta_0}u^{\alpha_0-\beta_0}e^{2t\alpha_0}(1+\mathcal{O}(u)), & z\to e^{-t}\\ e^{V(e^t)}\prod_{j=1}^m(1-z_je^{-t})^{2\alpha_j}e^{t(\alpha_j+\beta_j)}e^{-i\pi(\alpha_j+\beta_j)}z_j^{-(\alpha_j+\beta_j)} & \\ \times(1-e^{-2t})^{-(\alpha_0-\beta_0)}u^{\alpha_0+\beta_0}e^{-i\pi(\alpha_0+\beta_0)}(1+\mathcal{O}(u)), & z\to e^t.\end{cases} \tag{3.4.109}$$

For what follows, we only need the Δ_{12} element of the Δ_1 matrix and by (3.4.106) we have

$$\Delta_{12}=D(z)^2G^{-1}z^nf_t(z)^{-1}\Phi_{21}.$$

Putting all of this together with (3.4.97), we get the following as $z\to e^{-t}$,

$$\begin{aligned}(\Delta_1(z))_{12}=&\prod_{j=1}^m(1-z_j^{-1}e^{-t})^{2\beta_j}e^{-t(\alpha_j-\beta_j)}e^{-i\pi(\alpha_j-\beta_j)}z_j^{-(\alpha_j-\beta_j)}e^{-2i\pi\beta_0}\\ &\times(1-e^{-2t})^{\alpha_0+\beta_0}\exp\Big\{\sum_{k=0}^\infty V_ke^{-tk}\Big\}\exp\Big\{-\sum_{k=1}^\infty V_{-k}e^{tk}\Big\}\times n^{-2\beta_0}\\ &\times(2t)^{-(\alpha_0+\beta_0)}e^{t(\alpha_0+\beta_0)}\Phi_{21}(\lambda(z))(1+\mathcal{O}(u)),\end{aligned} \tag{3.4.110}$$

and for $z \to e^t$, we have $\Delta_{12} = D(z)^2 G^{-1} z^n f_t(z) \Phi_{21}$ and

$$(\Delta_1(z))_{12} = \prod_{j=1}^{m} (1 - z_j e^{-t})^{2\beta_j} e^{t(\alpha_j+\beta_j)} e^{-i\pi(\alpha_j+\beta_j)} z_j^{-(\alpha_j+\beta_j)} e^{-i\pi(\alpha_j+\beta_j)}$$
$$\times (1 - e^{-2t})^{-(\alpha_0-\beta_0)} \exp\Big\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \Big\} \exp\Big\{ \sum_{k=0}^{\infty} V_k e^{tk} \Big\} \times n^{-2\beta_0}$$
$$\times (2t)^{(\alpha_0-\beta_0)} e^{-2i\beta_0} e^{t(\alpha_0+\beta_0)} \Phi_{21}(\lambda(z))(1 + \mathcal{O}(u)). \tag{3.4.111}$$

Next, we need to determine an appropriate asymptotics for $\Phi(\lambda, x)$ both when x is small and when x becomes large. For the limit $\hat{\lambda} \to \infty$, we obtain two asymptotic expansions:

$$\Phi(\hat{\lambda}) = I + \frac{\hat{C}_1}{\hat{\lambda}} + \mathcal{O}(\hat{\lambda})^{-2}, \tag{3.4.112}$$

of which one holds when x is small, say, $0 < x < \delta$, and the other holds when $x > C$, where C is positive, and $\hat{\lambda} = \lambda(z) \pm \frac{x}{2}$. We aim to find a $\hat{C}_1$ for the expansion in (3.4.112) that works for the two cases.

Asymptotics for Φ for large x

In [13] when $x > C$ an expansion for the following function was obtained

$$\tilde{\Phi}(\xi) = \Phi(x\xi = \lambda; x). \tag{3.4.113}$$

The authors showed that

$$\tilde{C}_1 = \begin{pmatrix} \frac{x^{-2+2\alpha} e^{-x}}{\Gamma(\alpha_0-\beta_0)\Gamma(\alpha_0+\beta_0)} \big(1 + \mathcal{O}(x^{-1})\big) & -x^{-1+\alpha_0-\beta_0} e^{-x/2} \frac{e^{-2i\pi\beta_0}}{\Gamma(\alpha_0+\beta_0)} \big(1 + \mathcal{O}(x^{-1})\big) \\ x^{-1+\alpha_0-\beta_0} e^{-x/2} \frac{e^{2i\pi\beta_0}}{\Gamma(\alpha_0-\beta_0)} \big(1 + \mathcal{O}(x^{-1})\big) & \frac{-x^{-2+2\alpha} e^{-x}}{\Gamma(\alpha_0-\beta_0)\Gamma(\alpha_0+\beta_0)} \big(1 + \mathcal{O}(x^{-1})\big) \end{pmatrix}$$

when when $\xi \to \infty$. Further, by applying the transformation $\xi \to \frac{\lambda}{x}$, we obtain the following expansion for Φ_{21},

$$\Phi_{21}(\lambda; x) = x^{\alpha_0+\beta_0} e^{-x/2} \frac{e^{2\pi i\beta_0}}{\Gamma(\alpha_0-\beta_0)} \lambda^{-1} \big(1 + \mathcal{O}(x^{-1})\big) + \mathcal{O}(x^2/\lambda^2), \quad \text{as} \quad \lambda \to \infty \tag{3.4.114}$$

uniformly when $x > C$. Also, if we change $\lambda \to \lambda \pm \frac{x}{2}$, we get the following uniform asymptotics for $x > C$:

$$\Phi_{21}(\hat{\lambda}; x) = x^{\alpha_0+\beta_0} e^{-x/2} \frac{e^{2\pi i \beta_0}}{\Gamma(\alpha_0 - \beta_0)} (\hat{\lambda})^{-1}\big(1 + \mathcal{O}(x^{-1})\big) + \mathcal{O}(x^2/\hat{\lambda}^2), \quad \text{as} \quad \hat{\lambda} \to \infty. \tag{3.4.115}$$

Asymptotics for Φ for small x

To obtain the asymptotics of the Ψ Riemann-Hilbert problem and the Φ Riemann-Hilbert problem as $x \to 0$, we will use the analysis in [13]. Returning to the Ψ Riemann-Hilbert problem, we will examine the following function:

$$\tilde{\Psi}(\tilde{\lambda}; x) = e^{x\sigma_3/2} x^{-\beta_0\sigma_3} \times \begin{cases} \Psi_I(\frac{\lambda}{x} + 1, x) & \text{for} \quad \tilde{\lambda} \in I', \\ \Psi_{II}(\frac{\lambda}{x} + 1, x) & \text{for} \quad \tilde{\lambda} \in II', \\ \Psi_{III}(\frac{\lambda}{x} + 1, x) & \text{for} \quad \tilde{\lambda} \in III', \\ \Psi_{IV}(\frac{\lambda}{x} + 1, x) & \text{for} \quad \tilde{\lambda} \in IV', \\ \Psi_V(\frac{\lambda}{x} + 1, x) & \text{for} \quad \tilde{\lambda} \in V' \end{cases} \tag{3.4.116}$$

where $\tilde{\lambda}(z) = \lambda(z) - \frac{x}{2}$ and $\Psi_I, ..., \Psi_V$ are the analytic continuation of Ψ from $I, ..., V$ to $\mathbb{C} \setminus [0, \infty)$.

Here, the contour is turned into a contour in the $\tilde{\lambda}$-plane. In the ξ-plane, the contour is translated by half, and then by realising that $\xi(z) = \lambda(z)/x$. It is important to note that the point where the contour lines intersect is $\tilde{\lambda} = 0$ instead of $\xi = 1/2$.

Using the Riemann-Hilbert problem for Ψ, the following Riemann-Hilbert problem for $\tilde{\Psi}$ can be derived directly:

Riemann-Hilbert Problem for $\tilde{\Psi}$.

(RH-$\tilde{\Psi}$1): $\tilde{\Psi}$ is analytic for $\tilde{\lambda} \in \mathbb{C} \setminus \tilde{\Gamma}$. Where $\tilde{\Gamma} = \cup_{j=1}^{6} \tilde{\Gamma}_j$ and

$$\tilde{\Gamma}_1 = e^{i\pi/4}\mathbb{R}^+, \tilde{\Gamma}_2 = e^{3i\pi/4}\mathbb{R}^+, \tilde{\Gamma}_3 = e^{5i\pi/4}\mathbb{R}^+$$

$$\tilde{\Gamma}_4 = e^{7i\pi/4}\mathbb{R}^+, \tilde{\Gamma}_5 = \mathbb{R}^+, \tilde{\Gamma}_6 = (-x, 0).$$

(RH-$\tilde{\Psi}2$): $\tilde{\Psi}$ has continuous boundary values on $\tilde{\Gamma} \setminus \{-x, 0\}$ with the following jump conditions:

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) \begin{pmatrix} 1 & e^{i\pi(\alpha_0-\beta_0)} \\ 0 & 1 \end{pmatrix}, \quad \tilde{\lambda} \in \tilde{\Gamma}_1, \tag{3.4.117}$$

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) \begin{pmatrix} 1 & 0 \\ -e^{-i\pi(\alpha_0-\beta_0)} & 1 \end{pmatrix}, \quad \tilde{\lambda} \in \tilde{\Gamma}_2, \tag{3.4.118}$$

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) \begin{pmatrix} 1 & 0 \\ e^{i\pi(\alpha_0-\beta_0)} & 1 \end{pmatrix}, \quad \tilde{\lambda} \in \tilde{\Gamma}_3, \tag{3.4.119}$$

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) \begin{pmatrix} 1 & -e^{-i\pi(\alpha_0-\beta_0)} \\ 0 & 1 \end{pmatrix} \quad \tilde{\lambda} \in \tilde{\Gamma}_4, \tag{3.4.120}$$

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) e^{2\pi i \beta_0 \sigma_3} \quad \tilde{\lambda} \in \tilde{\Gamma}_5, \tag{3.4.121}$$

$$\tilde{\Psi}_+(\tilde{\lambda}) = \tilde{\Psi}_-(\tilde{\lambda}) e^{-\pi i(\alpha_0-\beta_0)\sigma_3} \quad \tilde{\lambda} \in \tilde{\Gamma}_6. \tag{3.4.122}$$

($\tilde{\Psi}3$): As $\tilde{\lambda} \to \infty$, it exhibits the following asymptotic behaviour:

$$\tilde{\Psi}(\tilde{\lambda}) = (I + \mathcal{O}(\tilde{\lambda}^{-1}))\tilde{\lambda}^{-\beta_0\sigma_3} e^{-\tilde{\lambda}\sigma_3/2}. \tag{3.4.123}$$

($\tilde{\Psi}4$): Near these points, it has the following behaviour:

$$\Psi(\tilde{\lambda}) = \mathcal{O} \begin{pmatrix} |\tilde{\lambda}+x|^{\frac{(\alpha_0-\beta_0)}{2}} & |\tilde{\lambda}+x|^{-\frac{(\alpha_0-\beta_0)}{2}} \\ |\tilde{\lambda}+x|^{\frac{(\alpha_0-\beta_0)}{2}} & |\tilde{\lambda}+x|^{-\frac{(\alpha_0-\beta_0)}{2}} \end{pmatrix} \quad \text{as} \quad \tilde{\lambda} \to -x, \tag{3.4.124}$$

$$\Psi(\tilde{\lambda}) = \mathcal{O} \begin{pmatrix} |\tilde{\lambda}|^{-\frac{(\alpha_0+\beta_0)}{2}} & |\tilde{\lambda}|^{\frac{(\alpha_0+\beta_0)}{2}} \\ |\tilde{\lambda}|^{-\frac{(\alpha_0+\beta_0)}{2}} & |\tilde{\lambda}|^{\frac{(\alpha_0+\beta_0)}{2}} \end{pmatrix} \quad \text{as} \quad \tilde{\lambda} \to 0, \tilde{\lambda} \in I', V'. \tag{3.4.125}$$

For the asymptotics in other sections, we can apply appropriate jump conditions in (3.4.125).

In [13], the $\tilde{\Psi}$ Riemann-Hilbert problem was solved by using the steepest descent method.

The authors constructed the global and local parametrix for small x, then matched them on the boundary U_ϵ of $\tilde{\lambda}=0$ that also contains $[-x,0]$. They obtained the following asymptotics for 21 element of $\Phi(\lambda;x)$:

$$\begin{aligned}\Phi_{21}(\tilde{\lambda}) &= e^{x/4}e^{2i\pi\beta_0}(\tilde{\lambda}+x)^{(\alpha_0-\beta_0)/2}\tilde{\lambda}^{(\alpha_0+\beta_0)/2}e^{\tilde{\lambda}/2}e^{x/4}e^{-\tilde{\lambda}/2}\\ &\quad\times\frac{\Gamma(1+\alpha_0+\beta_0)}{\Gamma(\alpha_0-\beta_0)}\tilde{\lambda}^{-\beta_0}\tilde{\lambda}^{-1}(1+\mathcal{O}(\tilde{\lambda}^{-1}))\\ &= e^{x/2}e^{2i\pi\beta_0}\frac{\Gamma(1+\alpha_0+\beta_0)}{\Gamma(\alpha_0-\beta_0)}\tilde{\lambda}^{-(\alpha_0-\beta_0)/2}(1+\frac{x}{\tilde{\lambda}})^{-(\alpha_0-\beta_0)/2}\tilde{\lambda}^{(\alpha_0-\beta_0)/2}\tilde{\lambda}^{-1}\\ &\quad\times(1+\mathcal{O}(\tilde{\lambda}^{-1}))\\ &= e^{x/2}e^{2i\pi\beta_0}\frac{\Gamma(1+\alpha_0+\beta_0)}{\Gamma(\alpha_0-\beta_0)}\tilde{\lambda}^{-1}(1+\mathcal{O}(\tilde{\lambda}^{-1})),\quad \text{as}\quad \tilde{\lambda}\to\infty\end{aligned} \tag{3.4.126}$$

uniformly for x sufficiently small. Also by translating $\tilde{\lambda}=\lambda'-x$ we obtain,

$$\Phi_{21}(\lambda') = e^{x/2}e^{2i\pi\beta_0}\frac{\Gamma(1+\alpha_0+\beta_0)}{\Gamma(\alpha_0-\beta_0)}\lambda'^{-1}\Big(1+\mathcal{O}(\lambda'^{-1})\Big)\quad \text{as}\quad \lambda'\to\infty, \tag{3.4.127}$$

uniformly for small x.

Now we introduce the following function as in [33]:

$$K(x) = e^{\frac{x}{2}}\int_x^\infty y^{\alpha_0+\beta_0}e^{-y}dy, \tag{3.4.128}$$

which has the following behaviour,

$$K(x)\sim\begin{cases} e^{-\frac{x}{2}}x^{\alpha_0+\beta_0} & \text{as}\quad x\to\infty\\ e^{\frac{x}{2}}\Gamma(1+\alpha_0+\beta_0) & \text{as}\quad x\to 0.\end{cases} \tag{3.4.129}$$

As a result, we may express $\Phi_{21}(\hat{\lambda};x)$ for a fixed x using $K(x)$, which holds true asymptotically for both large and small values of x. From (3.4.115), (3.4.126), and (3.4.127) we obtain

$$\Phi_{21}(\hat{\lambda};x) = \frac{e^{2i\pi\beta_0}}{\Gamma(\alpha_0-\beta_0)}K(x)(\hat{\lambda})^{-1}+\mathcal{O}\Big((\hat{\lambda})^{-2}\Big)\quad \text{as}\quad \hat{\lambda}\to\infty. \tag{3.4.130}$$

In addition, we observe that for t fixed, if $x = 2nt$, $\tilde{\lambda} = n \log \frac{z}{e^t}$ and $\lambda' = n \log \frac{z}{e^{-t}}$,

$$\frac{1}{\tilde{\lambda}} = \frac{e^t}{n(z - e^t)} + \mathcal{O}(z - e^t), \quad z \to e^t, \tag{3.4.131}$$

and

$$\frac{1}{\lambda'} = \frac{e^{-t}}{n(z - e^{-t})} + \mathcal{O}(z - e^{-t}), \quad z \to e^{-t}. \tag{3.4.132}$$

Then, we derive the expression for $(\Delta_1(z))_{12}$ by combining the results from (3.4.110), (3.4.111), and (3.4.126), (3.4.127) for $z \to e^{\pm t}$, respectively.

$$\begin{aligned}(\Delta_1(z))_{12} = \tilde{\lambda}^{-1} \prod_{j=1}^{m} &(1 - z_j^{-1} e^{-t})^{2\beta_j} e^{t(\alpha_j+\beta_j)} e^{-i\pi(\alpha_j-\beta_j)} z_j^{-(\alpha_j+\beta_j)} \\ &\times (1 - e^{-2t})^{\alpha_0+\beta_0} \exp\{\sum_{k=0}^{\infty} V_k e^{-tk}\} \exp\{-\sum_{k=1}^{\infty} V_{-k} e^{tk}\} \\ &\times (2t)^{\alpha_0\beta_0} e^{t(\alpha_0+\beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0 - \beta_0)} K(2nt).\end{aligned} \tag{3.4.133}$$

$$\begin{aligned}(\Delta_1(z))_{12} = \lambda'^{-1} \prod_{j=1}^{m} &(1 - z_j^{-1} e^{-t})^{2\beta_j} e^{-t(\alpha_j-\beta_j)} e^{-i\pi(\alpha_j-\beta_j)} z_j^{-(\alpha_j-\beta_j)} \\ &\times (1 - e^{-2t})^{\alpha_0+\beta_0} \exp\{\sum_{k=0}^{\infty} V_k e^{-tk}\} \exp\{-\sum_{k=1}^{\infty} V_{-k} e^{tk}\} \\ &\times (2t)^{-(\alpha_0+\beta_0)} e^{t(\alpha_0+\beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0 - \beta_0)} K(2nt).\end{aligned} \tag{3.4.134}$$

3.4.6 The small norm Riemann-Hilbert problem

Let us define the following matrix-valued function $R(z)$ of the final transformation for the original problem

$$R(z) = \begin{cases} S(z) N^{-1}(z), & z \in \mathbb{C} \setminus U_{z_j} \cup U_{z_0} \cup \Gamma, \\ S(z) P_{z_j}^{-1}(z), & z \in U_{z_j}, \\ S(z) P_{z_0}^{-1}(z), & z \in U_{z_0}. \end{cases} \tag{3.4.135}$$

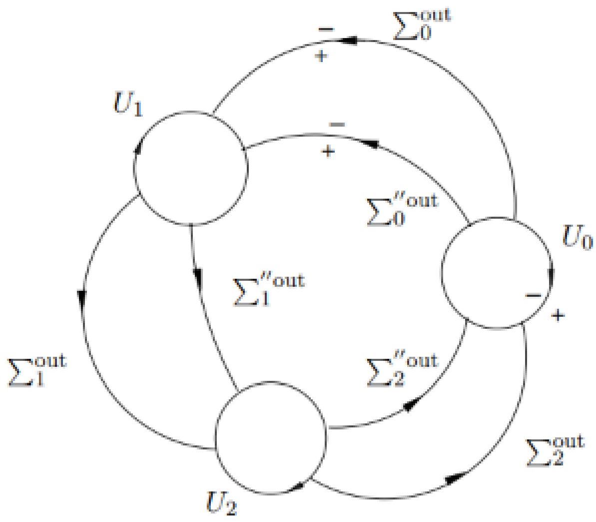

Figure 3.4: The Final RH problem

Remark 3.4.4. *So far, we have shown that the determinants of $P_{z_j}(z), P_{z_0}(z)$, and $N(z)$ are all one, and so they are invertible. Also, we have verified that $S(z)P_{z_j}^{-1}(z), j = 0, ..., m$ are analytic in U_{z_j} and that the jumps matrices of $S(z)$ agree with the function $N(z)$ on $\mathbb{C} \setminus U_{z_j} \cup U_{z_0} \cup \Gamma$.*

The following Riemann-Hilbert problem is solved by the function $R(z)$ using the preceding remark:

(RH-R1): $R(z)$ is analytic in $\mathbb{C} \setminus \Gamma \to \mathbb{C}^{2\times 2}$, where $\Gamma = \cup_{j=0}^{m} \partial U_{z_j} \cup_{j=0}^{m} \Sigma_j \cup_{j=0}^{m}$.

(RH-R2): $R(z)$ satisfies the jump condition

$$R_+(z) = R_-(z)V_R(z), \quad z \in \Gamma \tag{3.4.136}$$

with

$$V_R(z) = N(z)V_k(z)N(z)^{-1} \quad \text{with} \quad \begin{cases} z \in \cup_{j=0}^{m} \Sigma_j & \text{if} \quad k = 1 \\ z \in \cup_{j=0}^{m} \Sigma_j'' & \text{if} \quad k = 2, \end{cases} \tag{3.4.137}$$

where

$$V_1 = \begin{pmatrix} 1 & 0 \\ z^{-k} f_t(z)^{-1} & 1 \end{pmatrix}, \quad V_2 = \begin{pmatrix} 1 & 0 \\ z^{k} f_t(z)^{-1} & 1 \end{pmatrix}$$

and the jump conditions on ∂U_{z_0} and ∂U_{z_j} are given by

$$V_R(z) = \begin{cases} P_{z_j}(z) N^{-1}(z), & z \in \partial U_{z_j},\ j = 1, ..., m \\ P_{z_0}(z) N^{-1}(z), & z \in \partial U_{z_0}. \end{cases} \tag{3.4.138}$$

(RH-R3): The function behaves as follows as $z \to \infty$:

$$R(z) = I + \mathcal{O}(z^{-1}). \tag{3.4.139}$$

Remark 3.4.5. *The jump matrices on Σ and Σ'' in (3.4.137) behave in the way we require, that is, these matrices uniformly converge to the identity matrix at an exponential rate.*

$$R_+(z) = R_-(z)\Big(I + \mathcal{O}(e^{-\epsilon n})\Big) \quad \textit{as} \quad n \to \infty, \tag{3.4.140}$$

for a positive constant ϵ, with $0 < t < t_0$ and $x \notin \{x_1, ..., x_k\}$.

Regarding the jump matrix on the contour ∂U_{z_j}, where $j = 1, ..., m$, it can be observed that these matrices exhibit a uniform expansion in the inverse power of n which is further conjugated by $n^{\beta_j \sigma_3} z_j^{-n\sigma_3/2}$,

$$V_R(z) = I + \Delta_1(z) + \Delta_2(z) + ... + \Delta_k(z) + \Delta_{k+1}^{(r)}(z) \quad \text{for} \quad z \in \partial U_{z_j} \tag{3.4.141}$$

where $\Delta_k(z) = z_j^{n\sigma_3/2} n^{-\sigma_3 \beta_j} \mathcal{O}(n^{-1}) n^{\sigma_3 \beta_j} z_j^{-n\sigma_3/2} = \mathcal{O}(n^{2\max_j |\Re\beta_j| - k})$. Similarly, on the contour ∂U_{z_0},

$$P_{z_0}(z) N(z)^{-1} = I + \Delta_1(z) + ... + \Delta_k(z) + \Delta_{k+1}^{(r)}(z) \quad \text{for} \quad z \in \partial U_{z_0}. \tag{3.4.142}$$

The condition $V_R = I + o(1)$ is required to get at the solution, which implies that $\Re\beta_j$ and $\Re\beta_0$ are inside the interval $(-1/2, 1/2)$. Using (3.4.70) and (3.4.103), it can be observed that the jump matrices in (3.4.138) converge uniformly to the identity matrix $I + o(1)$. Consequently, we conclude that

$$R_+(z) = R_-(z)\Big(I + \mathcal{O}(n^{-1})\Big) \quad \text{as} \quad n \to \infty, \tag{3.4.143}$$

uniformly on ∂U_{z_j}.

Therefore, by using the asymptotic behaviour of the jump matrices in (3.4.140), and (3.4.143), we have

$$R(z) = I + \mathcal{O}(n^{-1}) \quad \text{as} \quad n \to \infty, \tag{3.4.144}$$

uniformly on $\mathbb{C} \setminus \Gamma = \cup_{j=0}^{m} \partial U_{z_j} \cup_{j=0}^{m} \Sigma_j \cup_{j=0}^{m} \Sigma_j''$.

Thus, the problem is a small norm problem in the case of the Fisher-Hartwig asymptotics, and the Neumann series provides its solution (see, Theorem 7.8 of [19]):

$$R(z) = I + \sum_{p=1}^{k} R_p(z) + R_{k+1}^{(r)}. \tag{3.4.145}$$

However, in the case of the Basor-Tracy conjecture when $|||\beta^{(t)}||| = 1$, the behaviour of $R(z)$ in (3.4.145) does not meet the requirements because $\Re\beta_j \notin (-1/2, 1/2)$. In order to address these particular cases, we will consider the Riemann-Hilbert problem for $R(z)$ as presented in [19]. Let

$$\hat{R}(z) = n^{\omega\sigma_3} R(z) n^{-\omega\sigma_3}, \tag{3.4.146}$$

where

$$\omega = \frac{1}{2}(\min_j \Re\beta_j + \max_j \Re\beta_j). \tag{3.4.147}$$

By this transformation, all $\Re\beta_j$ are moved into the strip $(-1/2, 1/2)$, which makes the asymptotics of the jump matrices above behave as $I + o(1)$.

Let us examine how the change in (3.4.146) affects the jump conditions of the Riemann-

Hilbert problem for $R(z)$:

$$\begin{aligned}\hat{R} &= n^{\omega\sigma_3}R_+(z)n^{-\omega\sigma_3} = n^{\omega\sigma_3}R_-(z)V_R(z)n^{-\omega\sigma_3} \\ &= n^{\omega\sigma_3}R_-(z)n^{-\omega\sigma_3}n^{\omega\sigma_3}V_R(z)n^{-\omega\sigma_3} \qquad . \\ &= \hat{R}_-n^{\omega\sigma_3}V_R(z)n^{-\omega\sigma_3}\end{aligned} \tag{3.4.148}$$

By this transformation, the jump matrices on the contours Σ and Σ'' have the same asymptotic behaviour as in the jump matrix for R in (3.4.140) but with different ϵ, $V_R(z) = I + \mathcal{O}(e^{-\epsilon n})$. The jump matrices on $\partial U_{z_j}, j = 0, ..., m$ have the form

$$\hat{R}_+ = \hat{R}_-\Big(I + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3} + +... + n^{\omega\sigma_3}\Delta_k(z)n^{-\omega\sigma_3} + n^{\omega\sigma_3}\Delta_{k+1}^{(r)}(z)n^{-\omega\sigma_3}\Big). \tag{3.4.149}$$

Each term is of order $\mathcal{O}(n^{2\max_j|\Re\beta_j-\omega|-k})$ and behaves asymptotically as $I + o(1)$.

Therefore, the Riemann-Hilbert problem for $\hat{R}$ is solvable for $\Re\beta_j \in (q-1/2, q+1/2)$, $j = 0, \ldots, m$, $q \in \mathbb{R}$, and the Neumann series

$$\hat{R}(z) = I + \sum_{p=1}^{k}\hat{R}_p(z) + \hat{R}_{k+1}^{(r)}(z) \tag{3.4.150}$$

provides the solution.

The function $\hat{R}_p(z)$ is repeatedly evaluated. The function $\hat{R}_p(z)$ is analytic outside of the boundary of $\cup_{j=0}^m \partial U_{z_j}$, $\hat{R}_p(z) \to 0$ at infinity for all p and satisfies

$$\hat{R}_{p,+}(z) = \hat{R}_{p,-}(z) + \sum_{i=1}^{p}\hat{R}_{p-i,-}(z)n^{\omega\sigma_3}\Delta_i(z)n^{-\omega\sigma_3}. \tag{3.4.151}$$

We set $\hat{R}_0(z) = I$. The Riemann-Hilbert problem for $\hat{R}_1(z)$ satisfies the following conditions:

(RH-$\hat{R}_1$1): $\hat{R}_1 : \mathbb{C} \setminus \cup_{j=0}^m \partial U_{z_j} \to \mathbb{C}^{2\times 2}$ is analytic

(RH-$\hat{R}_1$2): $\hat{R}_1(z)$ has the following jump conditions on the boundary of $\cup_{j=0}^m \partial U_{z_j}$,

$$\hat{R}_{1,+}(z) = \hat{R}_{1,-}(z) + n^{\omega\sigma_3}\Delta_1(z)n^{-\omega\sigma_3}. \tag{3.4.152}$$

(RH-$\hat{R}_1 3$): It has following behaviour as $z \to \infty$,

$$\hat{R}_1(z) \to 0. \tag{3.4.153}$$

We recall the transformation (3.4.146) to solve the problem, which gives us

$$R_p(z) = n^{-\omega\sigma_3} \hat{R}_p(z) n^{\omega\sigma_3}, \quad R_p^{(r)}(z) = n^{-\omega\sigma_3} \hat{R}_p^{(r)}(z) n^{\omega\sigma_3}. \tag{3.4.154}$$

Then, by applying the Plemelj formulas (see 2.6.2) and the residue theorem to this additive Riemann-Hilbert problem, we obtain:

$$\begin{aligned} R_1(z) &= \frac{1}{2\pi i} \int_{\partial U} \frac{\Delta_1(x) dx}{x - z} \\ &= \begin{cases} \sum_{k=1}^m \frac{A_k}{z - z_k} + \frac{A_{e^t}}{z - e^t} + \frac{A_{e^{-t}}}{z - e^{-t}} & z \in \mathbb{C} \setminus \cup_{j=0}^m U_{z_j} \\ \sum_{k=1}^m \frac{A_k}{z - z_k} + \frac{A_{e^t}}{z - e^t} + \frac{A_{e^{-t}}}{z - e^{-t}} - \Delta_1 & z \in U_{z_j}, \end{cases} \end{aligned} \tag{3.4.155}$$

where A_k, $A_{e^{\pm t}}$ are the coefficients in the Laurent expansion of

$$\Delta_1(z) = \frac{A_k}{z - z_k} + B_k + \mathcal{O}(z - z_k), \quad z \to z_k \quad k = 1, ..., m, \tag{3.4.156}$$

and

$$\Delta_1(z) = \frac{A_{e^{\pm t}}}{z - e^{\pm t}} + B_{e^{\pm t}} + \mathcal{O}(z - e^{\pm t}) \quad \text{as } z \to e^{\pm t}. \tag{3.4.157}$$

The coefficients A_k, and $A_{e^{\pm t}}$ are given below. The 12 entries of $\Delta_1(z)$ of each parametrix at z_j and $e^{\pm t}$ were computed by (3.4.74), (3.4.133), and (3.4.134), we compute the 12 elements of the matrices A_k, and $A_{e^{\pm t}}$ by using those in conjunction with (3.4.75), (3.4.131), and

(3.4.132), respectively. We have

$$
\begin{aligned}
A_k \equiv A_k^{(n)} = \frac{z_j}{n} \sum_{j=1}^{m} z_j^n e^{V_0} \exp\Big\{ \sum_{k=1}^{\infty} V_k z_j^k \Big\} \exp\Big\{ - \sum_{k=1}^{\infty} V_{-k} z_j^{-k} \Big\} (1 - z_j e^{-t})^{(\alpha_0+\beta_0)} \\
\times (1 - e^{-t} z_j^{-1})^{-(\alpha_0-\beta_0)} \times e^{t(\alpha_0+\beta_0)} \times \prod_{k\neq j} \left(\frac{z_j}{z_k} \right)^{\alpha_k} |z_j - z_k|^{2\beta_k} \\
\times \exp\Big\{ - i\pi \Big(\sum_{k=1}^{j-1} \alpha_k - \sum_{k=j+1}^{m} \alpha_k \Big) \Big\} n^{-2\beta_j} \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)} (1 + \mathcal{O}(u)),
\end{aligned}
\tag{3.4.158}
$$

$$
\begin{aligned}
A_{e^{-t}} = \frac{e^{-t}}{n} \prod_{j=1}^{m} e^{V_0} (1 - z_j^{-1} e^{-t})^{2\beta_j} (1 - e^{-2t})^{(\alpha_0+\beta_0)} \exp\Big\{ \sum_{k=1}^{\infty} V_k e^{-tk} \Big\} \exp\Big\{ - \sum_{k=1}^{\infty} V_{-k} e^{tk} \Big\} \\
\times e^{t(\alpha_0+\beta_0)} e^{-t(\alpha_j-\beta_j)} e^{-i\pi(\alpha_j-\beta_j)} z_j^{-(\alpha_j-\beta_j)} (2t)^{-(\alpha_0+\beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0-\beta_0)} K(2nt)(1+\mathcal{O}(n^{-1})),
\end{aligned}
\tag{3.4.159}
$$

and

$$
\begin{aligned}
A_{e^{t}} = \frac{e^{t}}{n} \prod_{j=1}^{m} e^{V_0} (1 - z_j e^{-t})^{2\beta_j} (1 - e^{-2t})^{-(\alpha_0-\beta_0)} \exp\Big\{ \sum_{k=1}^{\infty} V_k e^{tk} \Big\} \exp\Big\{ - \sum_{k=1}^{\infty} V_{-k} e^{-tk} \Big\} \\
\times e^{t(\alpha_0+\beta_0)} e^{t(\alpha_j+\beta_j)} e^{-i\pi(\alpha_j+\beta_j)} z_j^{-(\alpha_j+\beta_j)} (2t)^{(\alpha_0-\beta_0)} \frac{n^{-2\beta_0}}{\Gamma(\alpha_0-\beta_0)} K(2nt)(1+\mathcal{O}(n^{-1})).
\end{aligned}
\tag{3.4.160}
$$

We note that the coefficients B_k and $B_{e^{\pm t}}$ are easy to compute but are not needed in what follows.

Now we consider the Riemann-Hilbert problem for $\hat{R}_2(z)$:

(RH-$\hat{R}_2$1): $\hat{R}_2 : \mathbb{C} \setminus \cup_{j=0}^{m} \partial U_{z_j} \to \mathbb{C}^{2\times 2}$ is analytic.

(RH-$\hat{R}_2$2): It has the following jump conditions on $z \in \cup_{j=0}^{m} \partial U_{z_j}$,

$$
\hat{R}_{2,+}(z) = \hat{R}_{2,-}(z) + \hat{R}_{1,-}(z) \omega \sigma_3 \Delta_1(z) n^{-\omega\sigma_3} + \omega \sigma_3 \Delta_2(z) n^{-\omega\sigma_3}. \tag{3.4.161}
$$

(RH-$\hat{R}_2$3): As $z \to \infty$,

$$
\hat{R}_2(z) \to 0. \tag{3.4.162}
$$

By analysing the following integral and once again using the Plemelj formulas, we can show that

$$\hat{R}_2(z) = \frac{1}{2\pi i}\int_{\partial U}(\hat{R}_{1,-}(z)\omega\sigma_3\Delta_1(z)n^{-\omega\sigma_3} + \omega\sigma_3\Delta_2(z)n^{-\omega\sigma_3})\frac{dx}{x-z} \tag{3.4.163}$$

solves the preceding Riemann-Hilbert problem.

Further, by observing that each $\Delta_k(z) = \mathcal{O}(n^{2\max_j|\Re\beta_j|-k})$, similarly to [16], we get

$$\hat{R}^{(r)}_{k+1} = \mathcal{O}(|\hat{R}_{k+1}(z)|) + \mathcal{O}(|\hat{R}_{k+2}(z)|). \tag{3.4.164}$$

In particular,

$$R_3^{(r)}(z) = \begin{pmatrix} \mathcal{O}(\frac{\delta}{n}) + \mathcal{O}(\delta^2) & \mathcal{O}(\delta \max_k \frac{n^{-2\Re\beta_k}}{n}) \\ \mathcal{O}(\delta\max_k \frac{n^{2\Re\beta_k}}{n}) & \mathcal{O}(\frac{\delta}{n}) + \mathcal{O}(\delta^2), \end{pmatrix} \tag{3.4.165}$$

where

$$\delta = \max_{j,k} n^{2\Re(\beta_j-\beta_k-1)}. \tag{3.4.166}$$

3.5 The case of the Fisher-Hartwig asymptotics

We now go back to the OPs and prove Theorem 3.2.1 by retracing the steps $Y \to T \to S \to R$, and using the solution of the Riemann-Hilbert problem for $R(z)$. In the case of asymptotic behaviour for the Toeplitz determinant with respect to the symbol (3.1.1) if $|||\beta^{(t)}||| < 1$, we have the following uniform asymptotics for $z \in \mathbb{C}\setminus\Gamma$:

$$\hat{R}(z) = I + \mathcal{O}(n^{-1}) \quad \text{as} \quad n\to\infty. \tag{3.5.1}$$

We are only concerned with the solution near $z = e^t$ and $z = e^{-t}$ as required by the differential identity that we discuss next.

3.5.1 Differential identity

A differential identity will be used to connect Toeplitz determinants with the Riemann-Hilbert problem for orthogonal polynomials with respect to the symbol $f_t(z)$ as seen in the following result.

Lemma 3.5.1. *Let $t > 0$, $n \in \mathbb{N}$ and assume that the Riemann-Hilbert problem for $Y(z; n, t)$ is solvable with respect to $f_t(z)$. Then $D_n(f_t) \neq 0$ and the following differential identity holds:*

$$\frac{\partial}{\partial t}\log D_n(t) = -(\alpha_0+\beta_0)e^t(Y^{-1}\frac{dY}{dz})_{22}(e^t) + (\alpha_0-\beta_0)e^{-t}(Y^{-1}\frac{dY}{dz})_{22}(e^{-t}), \tag{3.5.2}$$

where $(Y^{-1}\frac{dY}{dz})_{22}(\xi)$ represents the 22 entry of the matrix obtained by multiplying the two matrices $Y^{-1}(z)$ and $\frac{dY}{dz}(z)$ (each entry Y is differentiated with respect to z) and evaluating the product at $z = \xi$.

In [34], the identity was proved by using orthogonal polynomials, whereas in [13] the identity has been proved by using Fredholm integral operators.

Using the reverse transformation, we have

$$\begin{aligned} Y(z) &= n^{-w\sigma_3}(I + \mathcal{O}(n^{-1}))n^{w\sigma_3}P_{z_0} \\ &= n^{-w\sigma_3}(I + \mathcal{O}(n^{-1}))n^{w\sigma_3}D(z)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}W(z)^{-1}\Phi(z)W(z), \end{aligned} \tag{3.5.3}$$

as $z \to e^{-t}$, and

$$\begin{aligned} Y(z) &= n^{-w\sigma_3}(I + \mathcal{O}(n^{-1}))n^{w\sigma_3}P_{z_0}z^{n\sigma_3} \\ &= n^{-w\sigma_3}(I + \mathcal{O}(n^{-1}))n^{w\sigma_3}D(z)^{\sigma_3}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}W(z)^{-1}\Phi(z)W(z) \end{aligned} \tag{3.5.4}$$

as $z \to e^t$.

Using the asymptotics for $\hat{R}(z)$ in (3.5.1), we observe that the the preceding two limits are uniform for $0 < t < t_0$ when $x = 2nt$ remains away from the set $\{x_0, x_1, ..., x_k\}$.

Using (3.5.3) and (3.5.4), we get

$$Y^{-1}Y_z' = \begin{cases} P^{-1}P_z' + P^{-1}(z)n^{-w\sigma_3}(I+\mathcal{O}(n^{-1}))^{-1}\mathcal{O}(n^{-1})_z' n^{w\sigma_3}P(z) & \text{if} \quad z \to e^{-t} \\ n^{-w\sigma_3}P^{-1}(z)n^{-w\sigma_3}(I+\mathcal{O}(n^{-1}))^{-1}\mathcal{O}(n^{-1})_z' n^{w\sigma_3}P(z)z^{n\sigma_3} & \\ +\frac{n\sigma_3}{z} + z^{-n\sigma_3}P^{-1}P_z' z^{n\sigma_3} & \text{if} \quad z \to e^{t} \end{cases} \tag{3.5.5}$$

and

$$P^{-1}P_z' = \begin{cases} \sigma_3\frac{A'}{A} + W^{-1}\Phi^{-1}\Phi_z' W - W^{-1}\Phi^{-1}\sigma_3\Phi W(\frac{A_z'}{A} + \frac{D_z'}{D}), & \text{near} \quad e^{-t} \\ -\sigma_3\frac{A'}{A} + W^{-1}\Phi^{-1}\Phi_z' W - W^{-1}\Phi^{-1}\sigma_3\Phi W(\frac{A_z'}{A} + \frac{D_z'}{D}), & \text{near} \quad e^{t}. \end{cases} \tag{3.5.6}$$

Define

$$A(z) = \begin{cases} -G(\lambda(z))^{-\frac{1}{2}}z^{\frac{n}{2}}f(z)^{-\frac{1}{2}}, & \text{for } |z| < 1 \\ G(\lambda(z))^{-\frac{1}{2}}z^{\frac{n}{2}}f(z)^{\frac{1}{2}}, & \text{for } |z| > 1 \end{cases} \tag{3.5.7}$$

and note that $W(z)$, which was defined in (3.4.98), can be rewritten as

$$W(z) = \begin{cases} A(z)^{\sigma_3}\sigma_3, & \text{for } |z| < 1 \\ A(z)^{\sigma_3}\sigma_1, & \text{for } |z| > 1. \end{cases} \tag{3.5.8}$$

The derivative of $A(z)$ can be expressed as follows:

$$A'(z) = \begin{cases} \frac{1}{2}(\frac{n}{z})\Big[-(\alpha_0-\beta_0)(n\log z + nt)^{-1} + (\alpha_0+\beta_0)(n\log z - nt)^{-1}\Big]A(z) & \\ -\frac{1}{2}f^{-1}(z)f'(z)A(z) & \text{near} \quad e^{-t} \\ \frac{1}{2}(\frac{n}{z})\Big[(\alpha_0-\beta_0)(n\log z + nt)^{-1} - (\alpha_0+\beta_0)(n\log z - nt)^{-1}\Big]A(z) & \\ +\frac{1}{2}f^{-1}(z)f'(z)A(z) & \text{near} \quad e^{t}, \end{cases} \tag{3.5.9}$$

where

$$G^{-\frac{1}{2}}(\lambda(z))' = \Big[(\alpha_0-\beta_0)(n\log z+nt)^{-1}-(\alpha_0+\beta_0)(n\log z-nt)^{-1}-1\Big] \times \frac{1}{2}(\frac{n}{z})G(\lambda(z))^{-\frac{1}{2}}, \tag{3.5.10}$$

and

$$f_z' = V'(z)f(z) - \sum_{j=1}^{m}(\alpha_j-\beta_j)z^{-1}f(z) + \sum_{j=1}^{m}2\alpha_j(z-z_j)^{-1}f(z) + (\alpha_0+\beta_0)(z-e^t)^{-1}f(z) + (\alpha_0-\beta_0)(z-e^{-t})^{-1}f(z) - (\alpha_0-\beta_0)z^{-1}f(z). \tag{3.5.11}$$

Next we compute the derivative of $|z-z_j|^{2\alpha_j}$ using the function $h_{\alpha_j}(z)$ in (3.4.19) as follows:

$$(h_{\alpha_j})'(z) = \alpha_j(z-z_j)^{-1}h_{\alpha_j}(z) - \frac{\alpha_j}{2}z^{-1}h_{\alpha_j}(z). \tag{3.5.12}$$

Consequently, by applying (3.5.7) and (3.5.9), we obtain

$$\frac{A'}{A}(z) = \begin{cases} \frac{\alpha_0+\beta_0}{4}e^{-t} + \frac{\alpha_0-\beta_0}{4}e^{-t}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) + \frac{1}{2}V_z'(e^t) - \sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}e^{-t} + \sum_{j=1}^{m}\alpha_j(e^t-z_j)^{-1} & \text{near} \quad e^t \\ \frac{\alpha_0-\beta_0}{4}e^{t} + \frac{\alpha_0+\beta_0}{4}e^{t}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) - \frac{1}{2}V_z'(e^{-t}) + \sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}e^{t} - \sum_{j=1}^{m}\alpha_j(e^{-t}-z_j)^{-1} & \text{near} \quad e^{-t} \end{cases} \tag{3.5.13}$$

and

$$\frac{D'}{D}(z) = \begin{cases} \sum_{j=1}^{m}(\alpha_j+\beta_j)\left(\frac{1}{e^{-t}-z_j}\right) - (\alpha_0+\beta_0)\frac{1}{2\sinh t} + \sum_{k=0}^{\infty}kV_ke^{-t(k-1)} & \text{for } z=e^{-t} \\ \sum_{j=1}^{m}(-\alpha_j+\beta_j)\left(\frac{z_je^{-t}}{e^t-z_j}\right) - (\alpha_0-\beta_0)\frac{e^{-2t}}{2sinht} - \sum_{k=-\infty}^{-1}kV_ke^{t(k-1)} & \text{for } z=e^{t} \end{cases} \tag{3.5.14}$$

Then by using (3.5.6), we will get the 22 entry on $(P^{-1}P')$ as follows:

$$(P^{-1}P')_{22}(e^{-t}) = -\frac{A'}{A}(e^{-t}) + (\Phi^{-1}\Phi_z')_{22}(e^{-t}) - (\Phi^{-1}\sigma_3\Phi)_{22}(e^{-t})\Big[\frac{A'}{A}(e^{-t}) + \frac{D'}{D}(e^{-t})\Big]$$

and

$$(P^{-1}P^{'})_{22}(e^{t})=\frac{A^{'}}{A}(e^{t})+(\Phi^{-1}\Phi_{z}^{'})_{11}(e^{t})-(\Phi^{-1}\sigma_{3}\Phi)_{11}(e^{t})\Big[\frac{A^{'}}{A}(e^{t})+\frac{D^{'}}{D}(e^{t})\Big].$$

Utilising the previous results and combining them with (3.5.6), we obtain

$$\begin{aligned}(e^{-t})(P^{-1}P^{'})_{22}(e^{-t})&=-\frac{\alpha_0-\beta_0}{4}-\frac{\alpha_0+\beta_0}{4}\Big(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\Big)+\frac{1}{2}e^{-t}V_{z}^{'}(e^{-t})-\sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}\\&+\sum_{j=1}^{m}\alpha_j\frac{e^{-t}}{(e^{-t}-z_j)}+e^{-t}(\Phi^{-1}\Phi_{z}^{'})_{22}(e^{-t})-(\Phi^{-1}\sigma_3\Phi)_{22}(e^{-t})\Big[\frac{\alpha_0-\beta_0}{4}\\&-\frac{1}{2}e^{-t}V_{z}^{'}(e^{-t})+\frac{\alpha_0+\beta_0}{4}\big(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\big)+\sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}\\&+\sum_{j=1}^{m}\beta_j\frac{e^{-t}}{e^{-t}-z_j}+\sum_{k=1}^{\infty}kV_ke^{-tk}\Big].\end{aligned} \tag{3.5.15}$$

For $z=e^{t}$ we obtain,

$$\begin{aligned}(e^{t})(P^{-1}P^{'})_{22}(e^{t})&=\frac{\alpha_0+\beta_0}{4}-\frac{\alpha_0-\beta_0}{4}\Big(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\Big)+\frac{1}{2}e^{t}V_{z}^{'}(e^{t})-\sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}\\&+\sum_{j=1}^{m}\alpha_j\frac{e^{t}}{(e^{t}-z_j)}+e^{t}(\Phi^{-1}\Phi_{z}^{'})_{11}(e^{t})-(\Phi^{-1}\sigma_3\Phi)_{11}(e^{t})\Big[\frac{\alpha_0+\beta_0}{4}\\&+\frac{\alpha_0-\beta_0}{4}\Big(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\Big)+\frac{1}{2}e^{t}V_{z}^{'}(e^{t})-\sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2}\Big(\frac{e^{t}+z_j}{2(e^{t}-z_j)}\Big)\\&+\sum_{j=1}^{m}\alpha_j\frac{e^{t}}{e^{t}-z_j}-\sum_{k=-\infty}^{-1}kV_ke^{tk}\Big].\end{aligned} \tag{3.5.16}$$

From (3.5.5) we obtain,

$$e^{-t}(Y^{-1}Y^{'})_{22}(e^{-t})=e^{-t}(P^{-1}P^{'})_{22}(e^{-t})+\big(\hat{\Phi^{-1}}(t)\mathcal{O}(n^{-1})\hat{\Phi}(t)\big)_{22},$$

where $n^{\beta\sigma_3}P(e^{t})=\hat{\Phi}(t)n^{\beta\sigma_3}$ and $\hat{\Phi}(t)$ is bounded in n as long as $\Phi(\frac{x}{2})$ is bounded. Then we find

$$
\begin{aligned}
e^{-t}(Y^{-1}Y')_{22}(e^{-t}) = &-\frac{\alpha_0-\beta_0}{4} - \frac{\alpha_0+\beta_0}{4}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) + \frac{1}{2}e^{-t}V_z'(e^{-t}) - \sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2} \\
&+ \sum_{j=1}^{m}\alpha_j\frac{e^{-t}}{(e^{-t}-z_j)} + e^{-t}(\Phi^{-1}\Phi_z')_{22}(e^{-t}) - (\Phi^{-1}\sigma_3\Phi)_{22}(e^{-t})\Bigg[\frac{\alpha_0-\beta_0}{4} \\
&- \frac{1}{2}e^{-t}V_z'(e^{-t}) + \sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2} + \frac{\alpha_0+\beta_0}{4}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) + \sum_{j=1}^{m}\beta_j\frac{e^{-t}}{e^{-t}-z_j} \\
&+ \sum_{k=1}^{\infty}kV_k e^{-tk}\Bigg] + \left(\hat{\Phi^{-1}}(t)\mathcal{O}(n^{-1})\hat{\Phi}(t)\right)_{22},
\end{aligned}
\tag{3.5.17}
$$

and near $z = e^t$, we have

$$
\begin{aligned}
e^{t}(Y^{-1}Y')_{22}(e^{t}) = &-n + \frac{\alpha_0+\beta_0}{4} + \frac{\alpha_0-\beta_0}{4}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) + \frac{1}{2}e^{t}V_z'(e^{t}) - \sum_{j=1}^{m}\frac{\alpha_j-\beta_j}{2} \\
&+ \sum_{j=1}^{m}\alpha_j\frac{e^{t}}{e^{t}-z_j)} + e^{t}(\Phi^{-1}\Phi_z')_{11}(e^{t}) - (\Phi^{-1}\sigma_3\Phi)_{11}(e^{t})\Bigg[\frac{\alpha_0+\beta_0}{4} \\
&+ \frac{\alpha_0-\beta_0}{4}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) + \frac{1}{2}e^{t}V_z'(e^{t}) - \sum_{j=1}^{m}(\alpha_j-\beta_j)\left(\frac{e^t+z_j}{2(e^t-z_j)}\right) \\
&+ \sum_{j=1}^{m}\alpha_j\frac{e^{t}}{e^{t}-z_j} - \sum_{k=-\infty}^{-1}kV_k e^{tk}\Bigg] + \left(\hat{\Phi^{-1}}(t)\mathcal{O}(n^{-1})\hat{\Phi}(t)\right)_{22}.
\end{aligned}
\tag{3.5.18}
$$

Using the differential identity (3.5.2), we obtain

$$\begin{aligned}
\frac{d}{dt}\log D_n(t) &= -(\alpha_0+\beta_0)e^t(Y^{-1}\frac{dY}{dz})_{22}(e^t) + (\alpha_0-\beta_0)e^{-t}(Y^{-1}\frac{dY}{dz})_{22}(e^{-t}) \\
&= n(\alpha_0+\beta_0) - \frac{\alpha_0^2+\beta_0^2}{2} - \frac{\alpha_0^2-\beta_0^2}{2}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) + \beta_0\sum_{j=1}^{m}(\alpha_j-\beta_j) \\
&- \frac{\alpha_0+\beta_0}{2}e^tV_z'(e^t) + \frac{\alpha_0-\beta_0}{2}e^{-t}V_z'(e^{-t}) + \sum_{j=1}^{m}\alpha_j\Bigg[-(\alpha_0+\beta_0)\frac{e^t}{e^t-z_j} \\
&+ (\alpha_0-\beta_0)\frac{e^{-t}}{e^{-t}-z_j}\Bigg] + 2wn + \frac{\alpha_0+\beta_0}{2}(\Phi^{-1}\sigma_3\Phi)_{11}(e^t)\Bigg[\frac{\alpha_0+\beta_0}{2} \\
&+ \frac{\alpha_0-\beta_0}{2}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) + e^tV_z'(e^t) - \sum_{j=1}^{m}(\alpha_j-\beta_j)\left(\frac{e^t+z_j}{(e^t-z_j)}\right) \\
&+ \sum_{j=1}^{m}2\alpha_j\frac{e^t}{e^t-z_j} - 2\sum_{k=-\infty}^{-1}kV_ke^{tk}\Bigg] - \frac{\alpha_0-\beta_0}{2}(\Phi^{-1}\sigma_3\Phi)_{22}\Bigg[\frac{\alpha_0-\beta_0}{2} \\
&+ \frac{\alpha_0+\beta_0}{2}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) - e^{-t}V_z'(e^{-t}) + \sum_{j=1}^{m}(\alpha_j-\beta_j) + \sum_{j=1}^{m}2\alpha_j\frac{e^{-t}}{e^{-t}-z_j} \\
&+ 2\sum_{k=1}^{\infty}kV_ke^{-tk}\Bigg] + (\hat{\Phi^{-1}}(t)\mathcal{O}(n^{-1})\hat{\Phi}(t))_{22},
\end{aligned}$$

where

$$w(x) = -\frac{(\alpha_0+\beta_0)}{2}(\Phi^{-1}\Phi_\lambda')_{11}(\frac{x}{2}) + \frac{(\alpha_0-\beta_0)}{2}(\Phi^{-1}\Phi_\lambda')_{22}(\frac{x}{2}). \tag{3.5.19}$$

We now recall some results on Painlevé V obtained in [13] that will be needed in what follows.

Proposition 3.5.2 (Proposition 4.4 of [13])**.** *Set*

$$a(\xi;x) = (\Psi(\xi;x)\sigma_3\Psi^{-1}(\xi;x))_{11} = -(\Psi(\xi;x)\sigma_3\Psi^{-1}(\xi;x))_{22}. \tag{3.5.20}$$

Then we have the following identities,

$$\frac{(\alpha_0-\beta_0)}{2}a(0;x) = A_{0,11} = -v(x) + \frac{(\alpha_0-\beta_0)}{2}, \tag{2.5.21}$$

$$\frac{(\alpha_0+\beta_0)}{2}a(1;x) = -A_{1,11} = -v(x) + \frac{(\alpha_0+\beta_0)}{2}. \tag{3.5.22}$$

Proposition 3.5.3 (Proposition 4.5, [13]). *Let $w(x)$ be defined as in (3.5.18). Then*

$$v(x) = -(xw(x))' \tag{3.5.23}$$

$$\sigma(x) = xw(x) \tag{3.5.24}$$

$$\sigma(x) = \int_x^\infty v(\xi)d\xi. \tag{3.5.25}$$

Now, applying the previous two propositions, we arrive at the following:

$$\begin{aligned}
\frac{d}{dt}\log D_n(t) &= n(\alpha_0+\beta_0) - \frac{\alpha_0^2+\beta_0^2}{2} - \frac{\alpha_0^2-\beta_0^2}{2}\left(\frac{1}{t}+\frac{e^{-t}}{\sinh t}\right) + \beta_0\sum_{j=1}^m(\alpha_j-\beta_j) \\
&- \frac{\alpha_0+\beta_0}{2}e^tV_z'(e^t) + \frac{\alpha_0-\beta_0}{2}e^{-t}V_z'(e^{-t}) + \sum_{j=1}^m \alpha_j\Big[-(\alpha_0+\beta_0)\frac{e^t}{e^t-z_j} \\
&+ (\alpha_0-\beta_0)\frac{e^{-t}}{e^{-t}-z_j}\Big] + \frac{1}{t}\sigma(x) + \left(-v(x)+\frac{\alpha_0+\beta_0}{2}\right)\Big\{\frac{\alpha_0+\beta_0}{2} \\
&+ \frac{\alpha_0-\beta_0}{2}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) - \sum_{j=1}^m(\alpha_j-\beta_j)\frac{e^t+z_j}{e^t-z_j} + \sum_{j=1}^m 2\alpha_j\frac{e^t}{e^t-z_j} \\
&+ \sum_{k=1}^\infty k(V_ke^{tk}+V_{-k}e^{-tk}\Big\} + \left(-v(x)+\frac{\alpha_0-\beta_0}{2}\right)\Big\{\frac{\alpha_0-\beta_0}{2} \\
&+ \frac{\alpha_0+\beta_0}{2}\left(\frac{1}{t}-\frac{e^{-t}}{\sinh t}\right) - \sum_{j=1}^m(\alpha_j-\beta_j) + \sum_{j=1}^m 2\beta_j\frac{e^{-t}}{e^{-t}-z_j}\sum_{k=1}^\infty k(V_ke^{-tk}+V_{-k}e^{tk}\Big\}.
\end{aligned}$$

The preceding expression can be simplified as follows:

$$\begin{aligned}
\frac{d}{dt}\log D_n(t) &= n(\alpha_0+\beta_0) - (\alpha_0^2-\beta_0^2)\left(\frac{e^{-t}}{\sinh t}\right) + (\alpha_0+\beta_0)\sum_{k=1}^\infty k(V_{-k}e^{tk}) + (\alpha_0-\beta_0)\sum_{k=1}^\infty k(V_ke^{-tk}) \\
&+ \frac{1}{t}\sigma(x) + \sum_{j=1}^m(\alpha_j+\beta_j)(\alpha_0-\beta_0)\left(\frac{e^{-t}}{e^{-t}-z_j}\right) - \sum_{j=1}^m(\alpha_j-\beta_j)(\alpha_0+\beta_0)\left(\frac{e^t}{e^t-z_j}\right) \\
&- \sum_{j=1}^m(\alpha_j-\beta_j)(\alpha_0+\beta_0) - v(x)\Big[\alpha_0 + \alpha_0(\frac{1}{t}-\frac{e^{-t}}{\sinh t}) + 2\sum_{k=1}^\infty k\cosh kt(V_k+V_{-k}) \\
&+ \sum_{j=1}^m 2\beta_j(\frac{e^t}{e^t-z_j}+\frac{e^{-t}}{e^{-t}-z_j})\Big] + \hat{\Phi}^{-1}(t)\mathcal{O}(n^{-1})\hat{\Phi}(t).
\end{aligned}$$

(3.5.26)

Using $\int_\epsilon^t \frac{d}{dt}\log D_n(\tau)d\tau = \log D_n(t) - \log D_n(\epsilon)$, where $0 < \epsilon < t < t_0$, and the following integrals

$$\int_\epsilon^t \frac{e^\tau}{e^\tau - z_j} d\tau = \log(1 - z_j e^{-t}) - \log(1 - z_j e^{-\epsilon}) + t - \epsilon. \tag{3.5.27}$$

$$\int_\epsilon^t \frac{e^{-\tau}}{e^{-\tau} - z_j} d\tau = -\log(\frac{z_j}{e^{i\pi}}) - \log(1 - z_j^{-1}e^{-t}) + \log(e^{-\epsilon} - z_j), \tag{3.5.28}$$

we obtain

$$\begin{aligned}
\log D_n(t) &= \log D_n(\epsilon) + n(\alpha_0 + \beta_0)(t-\epsilon) + \sum_{k=1}^\infty k[V_k - (\alpha_0+\beta_0)\frac{e^{-tk}}{k}][V_{-k} - (\alpha_0 - \beta_0)\frac{e^{-tk}}{k}] \\
&- \sum_{k=1}^\infty k V_k V_{-k} + (\alpha_0+\beta_0)\sum_{k=1}^\infty V_{-k}e^{-\epsilon k} + (\alpha_0 - \beta_0)\sum_{k=1}^\infty V_{-k}e^{-\epsilon k} + \Big((\alpha_0+\beta_0) \\
&- (\alpha_0+\beta_0)\Big)\sum_{j=1}^m (\alpha_j - \beta_j)(t-\epsilon) - (\alpha_0 - \beta_0)\sum_{j=1}^\infty (\alpha_j+\beta_j)\log(\frac{z_j}{e^{i\pi}}) \\
&+ (\alpha_0+\beta_0)\sum_{j=1}^\infty(\alpha_j - \beta_j)\sum_{k=1}^\infty(\frac{z_j^k e^{-tk}}{k}) + (\alpha_0 - \beta_0)\sum_{j=1}^\infty(\alpha_j+\beta_j)\sum_{k=1}^\infty(\frac{z_j^{-k}e^{-tk}}{k}) \\
&(\alpha_0+\beta_0)\sum_{j=1}^m(\alpha_j - \beta_j)\log(1 - z_j e^{-\epsilon}) + (\alpha_0 - \beta_0)\sum_{j=1}^m(\alpha_j+\beta_j)\log(e^{-\epsilon} - z_j) \\
&+ \Big[\int_{2n\epsilon}^{2nt}\frac{\sigma(x) - (\alpha_0^2 - \beta_0^2)}{x}dx + (\alpha_0^2 - \beta_0^2)\log(2nt) + (\alpha_0^2 - \beta_0^2)\log\left(\frac{n(1-e^{-2\epsilon})}{2n\epsilon}\right)\Big] \\
&- (\alpha_0^2 - \beta_0^2)\log n + R_n(t) + \mathcal{O}(n^{-1}),
\end{aligned} \tag{3.5.29}$$

where

$$R_n(t) = -\int_\epsilon^t v(2nt)\Big\{\alpha_0 + \alpha_0\Big(\frac{1}{t} - \frac{e^{-t}}{\sinh t}\Big) + \sum_{j=1}^m 2\beta_j\Big(\frac{e^t}{e^t - z_j} + \frac{e^{-t}}{e^{-t} - z_j}\Big)\Big\}. \tag{3.5.30}$$

In addition, as in [13], we have

$$|R_n(t)| < C\int_0^t |v(2nu)|du = \mathcal{O}(n^{-1}), \quad \text{as} \quad n\to\infty, \quad 0 < t < t_0. \tag{3.5.31}$$

Finally, after taking the limit $\epsilon \to 0$, using L'Hospital's rule for $\log\left(\frac{n(1-e^{-2\epsilon})}{2n\epsilon}\right)$ and performing some cancellations (recall that we take the branch of $\log z$ to be the negative real line), we obtain Theorem 3.2.1.

3.5.2 Emerging singularity at any point on the unit circle

Define $g_t(z) : \mathbb{T} \to \mathbb{C}$ by

$$g_t(z) = e^{V(z)}(z - e^{t+i\vartheta})^{\alpha+\beta}(z - e^{-t+i\vartheta})^{\alpha-\beta} z^{-\alpha+\beta} e^{-i\pi(\alpha+\beta)}. \tag{3.5.32}$$

Now we will consider the asymptotic behaviour of Toeplitz determinants $D_n(g_t(z))$. Note that for $t > 0$, this symbol is analytic in $\mathbb{C} \setminus ([0, e^{-t+i\vartheta_j}] \cup [e^{t+i\vartheta_j}, \infty])$. However, as $t \to 0$, the emerging singularity will appear at any point on the unit circle.

The Fourier coefficients of $\log g_t(z)$ can be easily computed and are given as follows:

$$(\log g_t)_0 = V_0 + (\alpha + \beta)(t + i\vartheta), \tag{3.5.33}$$

$$(\log g_t)_k = V_k - (\alpha + \beta)\frac{e^{-(t+i\vartheta)k}}{k}, \tag{3.5.34}$$

$$(\log g_t)_{-k} = V_{-k} - (\alpha - \beta)\frac{e^{-(t-i\vartheta)k}}{k}. \tag{3.5.35}$$

The asymptotic study of $D_n(g_t(z))$ can be split into two cases:

1. For $t > 0$, by Theorem 2.3.3, as $n \to \infty$,

$$\begin{aligned} \log D_n(g_t(z)) &= nV_0 + n(\alpha + \beta)(t + i\vartheta) \\ &+ \sum_{k=1}^{\infty} k\Big[V_{-k} - (\alpha - \beta)\frac{e^{-tk-i\vartheta k}}{k}\Big]\Big[V_k - (\alpha + \beta)\frac{e^{-tk+i\vartheta k}}{k}\Big] + o(1). \end{aligned} \tag{3.5.36}$$

2. For $t = 0$, by Theorem 2.4.2,

$$\begin{aligned} \log D_n(g_t(z)) &= nV_0 + n(\alpha + \beta)(i\vartheta) + \sum_{k=1}^{\infty} kV_k V_{-k} - (\alpha + \beta)\sum_{k=1}^{\infty} V_{-k} e^{i\vartheta k} \\ &- (\alpha - \beta)\sum_{k=1}^{\infty} V_k e^{-i\vartheta k} + (\alpha^2 - \beta^2)\log n + \log G_{\alpha+\beta,\alpha-\beta} + o(1). \end{aligned} \tag{3.5.37}$$

Note that (3.5.37) cannot be obtained from (3.5.36). To determine the asymptotics for Toeplitz determinants with respect to $g_t(z)$ that holds uniformly for $t \geq 0$, we apply the uniform asymptotic expansion for Toeplitz determinants in (2.7.2), using the Fourier coefficients for $g_t(z)$:

$$\log D_n(g_t(z)) = nV_0 + n(\alpha+\beta)(t+i\vartheta) + \sum_{k=1}^{\infty} k\Big[V_{-k} - (\alpha-\beta)\frac{e^{-tk-i\vartheta k}}{k}\Big]\Big[V_k - (\alpha+\beta)\frac{e^{-tk+i\vartheta k}}{k}\Big]$$
$$+ \log G_{\alpha+\beta,\alpha-\beta} + \Omega(2nt) + o(1), \tag{3.5.38}$$

where $G(z)$ is the Barnes G-function (2.4.12), and $\Omega(2nt)$ is defined by

$$\tilde{\Omega}(2nt) = \exp\Omega(2nt) = \exp\left\{\int_0^{2nt} \frac{\sigma(x) - \alpha_0^2 + \beta_0^2}{x}dx + (\alpha^2 - \beta_0^2)\log 2nt\right\}. \tag{3.5.39}$$

Recall that the function $\sigma(x)$, defined in (2.7.4), is related to Painlevé V transcendent.

Reconstructing Szegő and Fisher-Hartwig asymptotics

We can find the asymptotics in (3.5.36) and (3.5.37) from the formula in (3.5.38). To get Fisher-Hartwig asymptotics, we let $t \to 0$ and n be fixed. Next, we look at the function $\Omega(2nt)$ in (2.7.4), which yields the following asymptotics:

$$\Omega(2nt) = \left(\alpha^2 - \beta^2\right)\log(2nt) + o(1), \tag{3.5.40}$$

in addition by applying the formula $\sum_{k=1}^{\infty} \frac{e^{-2kt}}{k} = -\log(1 - e^{-2t})$, then (3.5.37) provides the asymptotics.

The expansion (3.5.38) should also match the Szegő asymptotics for fixed t. We can easily observe that for a fixed t, the $\mathcal{O}(n)$-term produces the same term in the Szegő asymptotics. The similarity of the $\mathcal{O}(1)$-terms, on the other hand, leads to an interesting identity involving the Painlevé function $\sigma(x)$ via (3.5.40),

$$\Omega(+\infty) = -\log G_{\alpha+\beta,\alpha-\beta} = -\log\frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)}. \tag{3.5.41}$$

3.6 The case of Basor-Tracy conjecture

Consider the asymptotic behaviour of the Toeplitz determinant $D_n(f_t(z))$ when the seminorm $|||\beta^{(t)}||| = \max_{1\leq j,k\leq m} |\Re\beta_j - \Re\beta_k| = 1$. This means that $\Re\beta_0 = q - 1/2$ and $\Re\beta_j = q + 1/2$ can be expressed for some $q \in \mathbb{R}$, and $j \in \mathbb{N}$. To determine this asymptotics of $D_n(f_t(z))$ as $n \to \infty$ and $t \to 0$, we will investigate two possible cases:

1. If $|||\beta_1, ..., \beta_m||| < 1$ for $t > 0$ and at emerging singularity $|||\beta_0, ..., \beta_m||| = 1$
2. If $|||\beta_1, ..., \beta_m||| = 1$ for $t > 0$ and at the emerging singularity $|||\beta_0, ..., \beta_m||| = 1$.

Remark 3.6.1. *Note that if there are no jumps or less than two jump singularities, the Tracy-Basor conjecture will not be needed. Thus, we need to consider the case when $|||\beta^{(t)}||| = 1$ because our symbol in (3.1.1) possesses m Fisher-Hartwig singularities when $t > 0$ and $m+1$ Fisher-Hartwig singularities when $t = 0$. For $|||\beta^{(t)}||| \geq 1$ when $0 < t < t_0$, this could be addressed in the same way as in [16].*

3.6.1 The first case

Using Lemma 3.6.3 and some methods described in [16], we can deal with the case when the semi-norm equals one.Without loss of generality, and by relabeling β_j according to the increasing real component, we assume that

$$\Re\beta_0 < \Re\beta_1 \leq ... \leq \Re\beta_{m-1} < \Re\beta_m. \tag{3.6.1}$$

Next, we introduce the symbol $f_t(z)$ as defined in (3.1.1), but replace the β_j parameters by $\tilde{\beta}_j$, $j = 0, ..., m$ as follows:

1. $\tilde{\beta}_0 = \beta_0$,
2. $\tilde{\beta}_j = \beta_j$ for $j = 1, ..., m-1$,
3. $\tilde{\beta}_m = \beta_m - 1$.

The symbol $\tilde{f}_t(z)$ satisfies $|||\tilde{\beta}_0, \tilde{\beta}_1, ..., \tilde{\beta}_m||| < 1$ and the asymptotic behaviour of Toeplitz determinants was computed in the previous section with β_j replaced by $\tilde{\beta}_j$. Subsequently, our goal is to establish a link between the two symbols, namely the original symbol with $|||\beta^{(t)}||| = 1$ and $\tilde{f}(z,t)$, by utilizing the known asymptotics properties of $\tilde{f}(z,t)$.

Remark 3.6.2. *If (3.6.1) holds, we can define a non-trivial Fisher-Hartwig representation by shifting β_0 by $+1$. Then, we can connect the symbol $\tilde{f}_t(z)$ by the symbol $\hat{f}_t(z)$ by the following relation:*

$$\hat{f}_t(z) = (-1)\frac{z(z-e^t)}{(z-e^{-t})}\tilde{f}_t(z) \tag{3.6.2}$$

where $\hat{f}_t(z)$ satisfies the following

$$\begin{aligned} f_t(z) &= e^{V(z)}(z-e^t)^{\alpha_0+\beta_0+1-1}(z-e^{-t})^{\alpha_0-\beta_0+1-1}z^{-\alpha_0+\beta_0+1-1}e^{-i\pi(\alpha_0+\beta_0+1-1)} \\ &\times z^{\sum_{j=1}^{m-1}\beta_j}z^{\beta_m-1+1}\times\prod_{1<j<k<m-1}|z_j-z_k|^{2\alpha_j}g_{z_j,\beta_j}z_j^{-\beta_j}\times|z_m-z_k|^{2\alpha_m} \\ &\times g_{z_m,\beta_m-1+1}(z)z_m^{-\beta_m+1-1} \\ &= \frac{(z-e^{-t})}{(z-e^t)}z_m^{-1}\hat{f}(z,t) \end{aligned} \tag{3.6.3}$$

with parameters $\hat{\beta_0} = \beta_0+1$, $\hat{\beta_j} = \beta_j$, $j = 1,...,m-1$, and $\hat{\beta_m} = \beta_m - 1$.

Additionally, we can change $\tilde{\beta}_j$ in $\tilde{f}_t(z)$ back by $+1$; this is referred to as a trivial Fisher-Hartwig representation. Then we can use the following relation:

$$f(z,t) = (-1)zz_m^{-1}\tilde{f}(z,t) \tag{3.6.4}$$

Lemma 3.6.3 (Lemma 2.4 of [16])**.** *Let $D_n(f)$ with respect to the symbol $f(z)$ be nonzero for all $n \geq N_0$ with fixed $N_0 \geq 0$. Let $\Phi_k = \frac{\phi_k(z)}{\chi_k}$, $\hat{\Phi}_k = \frac{\hat{\phi}_k(z)}{\chi_k}$, with $k = N_0, N_0+1, \ldots$, be the system of monic polynomials orthogonal on the unit circle with the weight $f(z)$. Fix $l > 0$. Thus if*

$$F_n = \begin{vmatrix} \Phi_k(0) & \Phi_{k+1}(0) & \ldots & \Phi_{k+l-1}(0) \\ \frac{d}{dz}\Phi_k(0) & \frac{d}{dz}\Phi_{k+1} & \ldots & \frac{d}{dz}\Phi_{k+l-1}(0) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{d^{l-1}}{dz^{l-1}}\Phi_k(0) & \frac{d^{l-1}}{dz^{l-1}}\Phi_{k+1}(0) & \ldots & \frac{d^{l-1}}{dz^{l-1}}\Phi_{k+l-1}(0) \end{vmatrix} \neq 0 \tag{3.6.5}$$

we have

$$D_n(z^l f(z)) = \frac{(-1)^{ln} F_n}{\prod_{j=1}^{l-1} j!} D_n(f(z)), \quad n \geq N_0. \tag{3.6.6}$$

In particular, if $l = 1$, $\phi_k(0) \neq 0, k = N_0, N_0 + 1, ..., n - 1$, *we have*

$$D_n(zf(z)) = (-1)^n \frac{\phi_n(0)}{\chi_n} D_n(f(z)), \quad n \geq N_0. \tag{3.6.7}$$

Thus, using (3.6.7) and the relation in (3.6.4), we have

$$\begin{aligned} D_n\Big((-1)zz_m^{-1}\tilde{f}(z,t)\Big) &= z_m^{-n} D_n(z\tilde{f}_t(z)) \\ &= z_m^{-n} \frac{\phi_n(0)}{\chi_n} D_n(\tilde{f}_t(z)) \end{aligned} \tag{3.6.8}$$

If Y is the solution of Riemann-Hilbert problem with respect to the symbol $\tilde{f}_t(z)$ in (3.3.1), then

$$Y_{11}(0) = \chi_n^{-1}\phi_n(0). \tag{3.6.9}$$

Using the final solution of Riemann-Hilbert problem $R(z)$ and transformations $R(z) \mapsto S(z) \mapsto T(z) \mapsto Y(z)$, we get for $|z| < 1$

$$Y(z) = \Big[I + R_1(z) + R_2(z) + R_3^{(r)}(z)\Big] D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{3.6.10}$$

which subsequently gives $R_{12}(0)$,

$$\frac{\phi_n(0)}{\chi_n} = Y_{11}(0) = -D(0)^{-1}\Big[R_{1,12}(0) + R_{2,12}(0) + \mathcal{O}\big(\delta \max_k \frac{n^{-2\Re\beta_k}}{n}\big)\Big]. \tag{3.6.11}$$

Moreover, by (3.4.13), we have

$$D(0) = e^{V_0} e^{t(\alpha_0 + \tilde{\beta}_0)} \tag{3.6.12}$$

and so

$$\frac{\phi_n(0)}{\chi_n} = D(0)^{-1}\Big(\sum_{j=1}^{m}\frac{A^n_{j,12}}{z_j} + \frac{A_{e^t,12}}{e^t} + \frac{A_{e^{-t},12}}{e^{-t}} + \mathcal{O}\big(\delta \max_k \frac{n^{-2\Re\beta_k}}{n})\Big). \tag{3.6.13}$$

Using (3.4.13), (3.4.158), (3.4.159), and (3.4.160), we obtain the asymptotics of OPs with respect to $\tilde{f}_t(z)$ as follows:

$$\begin{aligned}\frac{\phi_n(0)}{\chi_n} &= \Big[\sum_{j=1}^{m} z_j^n \exp\Big\{\sum_{k=1}^{\infty}[V_k - (\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}]z_j^k\Big\}\exp\Big\{-\sum_{k=1}^{\infty}[V_{-k} - (\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}]z_j^{-k}\Big\}\\ &\quad\times n^{-2\tilde{\beta}_j-1}\frac{\Gamma(1+\alpha_j+\tilde{\beta}_j)}{\Gamma(\alpha_j-\tilde{\beta}_j)}\times\nu_j + \mathcal{O}\Big(\Big[\delta+\frac{1}{n}\Big]n^{-2\Re\tilde{\beta}_j-1}\Big) + \prod_{j=1}^{m}(1-z_je^{-t})^{2\tilde{\beta}_j}\\ &\quad\times(1-e^{-2t})^{-(\alpha_0-\tilde{\beta}_0)}\times\exp\Big\{\sum_{k=1}^{\infty}V_ke^{tk}\Big\}\times\exp\Big\{-\sum_{k=1}^{\infty}V_{-k}e^{-tk}\Big\}\times(e^{-i\pi}e^tz_j^{-1})^{(\alpha_j+\tilde{\beta}_j)}\\ &\quad\times(2t)^{\alpha_0-\tilde{\beta}_0}\frac{n^{-2\tilde{\beta}_0-1}}{\Gamma(\alpha_0-\tilde{\beta}_0)}K(2nt) + \prod_{j=1}^{m}(1-z_j^{-1}e^{-t})^{2\tilde{\beta}_j}\times(1-e^{-2t})^{(\alpha_0+\tilde{\beta}_0)}\times\exp\Big\{\sum_{k=1}^{\infty}V_ke^{-tk}\Big\}\\ &\quad\times\exp\Big\{-\sum_{k=1}^{\infty}V_{-k}e^{tk}\Big\}(e^{-i\pi}e^{-t}z_j^{-1})^{(\alpha_j-\tilde{\beta}_j)}\times(2t)^{-(\alpha_0+\tilde{\beta}_0)}\frac{n^{-2\tilde{\beta}_0-1}}{\Gamma(\alpha_0-\tilde{\beta}_0)}K(2nt)\Big](1+o(1)),\end{aligned} \tag{3.6.14}$$

where $\nu_j = \prod_{k\neq j}\left(\frac{z_j}{z_ke^{i\pi}}\right)^{\alpha_k}|z_j-z_k|^{2\tilde{\beta}_k}$.

Using Lemma 3.6.3, we have

$$D_n(f_t(z)) = z_m^{-n}\times\Big\{D(0)^{-1}\Big(\frac{A^n_{m,12}}{z_m} + \frac{A_{e^t,12}}{e^t} + \frac{A_{e^{-t},12}}{e^{-t}} + \mathcal{O}\big(\delta\max_k\frac{n^{-2\Re\beta_k}}{n})\Big)\Big\}D_n(\tilde{f}_t(z)) \tag{3.6.15}$$

After a simple computation, and by using the following expression

$$\exp\Big\{\log(1-z)^{\alpha\pm\beta}\Big\} = \exp\Big\{-(\alpha\pm\beta)\sum_{k=1}^{\infty}\frac{z^k}{k}\Big\}, \qquad \text{for} \quad |z|<1.$$

In addition, based on the properties and definition of the Barnes G-function, the following equation was used:

$$G_{\alpha_j+\beta_j+1,\alpha_j-\beta_j-1} = \frac{\Gamma(1+\alpha_j+\beta_j)}{\Gamma(\alpha_j-\beta_j)}G_{\alpha_j+\beta_j,\alpha_j-\beta_j}. \tag{3.6.16}$$

we conclude that,

$$
\begin{aligned}
D_n(f_t(z)) = D_n\bigg(f(z;\alpha_0,\alpha_j,\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_m+1)\bigg) \times \tilde{\Omega}(2nt)(1+o(1)) \\
+ z_m^{-n}\bigg[\exp\bigg\{nV_0 + nt(\alpha_0+\tilde{\beta}_0)\bigg\}\exp\bigg\{\sum_{k=1}^{\infty} k\Big[V_k - (\alpha_0+\tilde{\beta}_0+1)\frac{e^{-kt}}{k}\Big] \\
\times \Big[V_{-k} - (\alpha_0-\tilde{\beta}_0)\frac{e^{-kt}}{k}\Big]\bigg\} \times n^{\sum_{j=1}^{m}(\alpha_j^2-\tilde{\beta}_j{}^2)} \times n^{-2\tilde{\beta}_0-1} \times \exp\bigg\{-\sum_{j=1}^{m-1}(\alpha_j-\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k \\
-(\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^k\bigg\} \times \exp\bigg\{-(\alpha_m-\tilde{\beta_m})\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\tilde{\beta}_0+1)\frac{e^{-tk}}{k}\Big]z_m^k\bigg\} \\
\times (1-e^t z_m^{-1})^{\alpha_m+\tilde{\beta_m}} \times \exp\bigg\{-\sum_{j=1}^{m}(\alpha_j+\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\bigg\} \\
\times \prod_{j=1}^{m} G_{\alpha_j+\tilde{\beta}_j,\alpha_j-\tilde{\beta}_j} \times \frac{G_{\alpha_0+\tilde{\beta}_0+1,\alpha_0-\tilde{\beta}_0-1}}{\Gamma(1+\alpha_0+\tilde{\beta}_0)} \times (2t)^{\alpha_0-\tilde{\beta}_0} \times \exp\bigg\{\sum_{k=1}^{\infty}V_k e^{tk}\bigg\} \\
\times \prod_{1\le j\le k<m} |z_j-z_k|^{2\left(\beta_k\tilde{\beta}_j-\alpha_j\alpha_k\right)}\left(\frac{z_k}{z_j e^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\tilde{\beta}_j\right)} \times k(2nt) \times \tilde{\Omega}(2nt) \\
+ \exp\bigg\{nV_0+nt(\alpha_0+\tilde{\beta}_0)\bigg\} \times \exp\bigg\{\sum_{k=1}^{\infty}k\Big[V_k-(\alpha_0+\tilde{\beta}_0)\frac{e^{-kt}}{k}\Big]\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0-1)\frac{e^{-kt}}{k}\Big]\bigg\} \\
\times \exp\bigg\{-\sum_{j=1}^{m}(\alpha_j-\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^k\bigg\}\exp\bigg\{-\sum_{j=1}^{m-1}(\alpha_j+\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k} \\
-(\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\bigg\} \times \exp\bigg\{-(\alpha_m+\tilde{\beta_m})\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0-1)\frac{e^{-tk}}{k}\Big]z_m^{-k}\bigg\} \\
\times \exp\bigg\{-\sum_{k=1}^{\infty}V_{-k}e^{tk}\bigg\} \times \prod_{j=1}^{m}G_{\alpha_j+\tilde{\beta}_j,\alpha_j-\tilde{\beta}_j} \times \frac{G_{\alpha_0+\tilde{\beta}_0+1,\alpha_0-\tilde{\beta}_0-1}}{\Gamma(1+\alpha_0+\tilde{\beta}_0)} \times n^{\sum_{j=1}^{m}(\alpha_j^2-\tilde{\beta}_j{}^2)}n^{-2\tilde{\beta}_0-1} \\
\times (2t)^{-(\alpha_0+\tilde{\beta}_0)} \times (1-e^t z_m)^{-(\alpha_m-\tilde{\beta_m})} \times k(2nt) \times \prod_{1\le j\le k<m}|z_j-z_k|^{2\left(\beta_k\tilde{\beta}_j-\alpha_j\alpha_k\right)} \\
\times \left(\frac{z_k}{z_j e^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\tilde{\beta}_j\right)} \times \tilde{\Omega}(2nt)\bigg]\bigg(1+o(1)\bigg).
\end{aligned}
\tag{3.6.17}
$$

Recalling the relation between β_j, $\tilde{\beta}_j$, and $\hat{\beta}_j$ for $j = 0, ..., m$, we get

$$
\begin{aligned}
D_n(f_t(z)) = D_n\Big(f(z;\alpha_0,\alpha_j,\alpha_m,\beta_0,\beta_j,\beta_m)\Big) \times \tilde{\Omega}(2nt)(1+o(1)) \\
+ z_m^{-n}\Big[\exp\Big\{nV_0 + nt(\alpha_0+\beta_0)\Big\}\exp\Big\{\sum_{k=1}^{\infty} k\Big[V_k - (\alpha_0+\hat{\beta}_0)\frac{e^{-kt}}{k}\Big] \\
\times \Big[V_{-k} - (\alpha_0-\beta_0)\frac{e^{-kt}}{k}\Big]\Big\} \times n^{\sum_{j=1}^{m}(\alpha_j^2-\hat{\beta}_j^2)} \times n^{-2\beta_0-1} \times \exp\Big\{-\sum_{j=1}^{m-1}(\alpha_j-\hat{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k \\
-(\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big]z_j^k\Big\} \times \exp\Big\{-(\alpha_m-\hat{\beta}_m)\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\hat{\beta}_0)\frac{e^{-tk}}{k}\Big]z_m^k\Big\} \\
(1-e^t z_m^{-1})^{\alpha_m+\hat{\beta}_m} \times \exp\Big\{-\sum_{j=1}^{m}(\alpha_j+\hat{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\} \\
\times \prod_{j=1}^{m} G_{\alpha_j+\hat{\beta}_j,\alpha_j-\hat{\beta}_j} \times \frac{G_{\alpha_0+\hat{\beta}_0,\alpha_0-\hat{\beta}_0}}{\Gamma(1+\alpha_0+\beta_0)} \times (2t)^{\alpha_0-\beta_0} \times \exp\Big\{\sum_{k=1}^{\infty} V_k e^{tk}\Big\} \\
\times \prod_{1\le j\le k<m} |z_j-z_k|^{2\left(\beta_k\hat{\beta}_j-\alpha_j\alpha_k\right)}\left(\frac{z_k}{z_j e^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\hat{\beta}_j\right)} \times k(2nt)\times\tilde{\Omega}(2nt) \\
+ \exp\Big\{nV_0+nt(\alpha_0+\beta_0)\Big\}\exp\Big\{\sum_{k=1}^{\infty}k\Big[V_k-(\alpha_0+\beta_0)\frac{e^{-kt}}{k}\Big]\Big[V_{-k}-(\alpha_0-\hat{\beta}_0)\frac{e^{-kt}}{k}\Big]\Big\} \\
\times \exp\Big\{-\sum_{j=1}^{m}(\alpha_j-\hat{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\beta_0)\frac{e^{-tk}}{k}\Big]z_j^k\Big\}\exp\Big\{-\sum_{j=1}^{m-1}(\alpha_j+\hat{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k} \\
-(\alpha_0-\beta_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\}\times\exp\Big\{-(\alpha_m+\hat{\beta}_m)\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\hat{\beta}_0)\frac{e^{-tk}}{k}\Big]z_m^{-k}\Big\} \\
\times \exp\Big\{-\sum_{k=1}^{\infty}V_{-k}e^{tk}\Big\}\times\prod_{j=1}^{m}G_{\alpha_j+\hat{\beta}_j,\alpha_j-\hat{\beta}_j}\times\frac{G_{\alpha_0+\hat{\beta}_0,\alpha_0-\hat{\beta}_0}}{\Gamma(1+\alpha_0+\beta_0)}\times n^{\sum_{j=1}^{m}(\alpha_j^2-\hat{\beta}_j^2)}n^{-2\beta_0-1} \\
\times (2t)^{-(\alpha_0+\beta_0)}\times(1-e^t z_m)^{-(\alpha_m-\hat{\beta}_m)}\times k(2nt)\times\prod_{1\le j\le k<m}|z_j-z_k|^{2\left(\beta_k\hat{\beta}_j-\alpha_j\alpha_k\right)} \\
\times\left(\frac{z_k}{z_j e^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\hat{\beta}_j\right)}\times\tilde{\Omega}(2nt)\Big]\Big(1+o(1)\Big)
\end{aligned}
\tag{3.6.18}
$$

By Simplifying the expression (3.6.17), we obtain

$$\begin{aligned} D_n(f_t(z)) &= R\Big(f(z;\beta_0,\beta_j,\beta_m)\Big) \times \tilde{\Omega}(2nt)(1+o(1)) \\ &+ \Big(z_m^{-n}\Big)^n \times \tilde{\Omega}(2nt) \times \frac{K(2nt)}{e^{nt}} \times \frac{n^{-2\beta_0-1}}{\Gamma(1+\alpha_0+\beta_0)} \\ &\times (1-e^{-2t})^{-2\beta_0-1}\Sigma'(t) \times R\Big(f(z;\hat{\beta}_0,\hat{\beta}_j,\hat{\beta}_m)\Big)\Big(1+o(1)\Big) \end{aligned} \tag{3.6.19}$$

where $R\Big(f(z;\hat{\beta}_0,\hat{\beta}_j,\hat{\beta}_m)\Big)$ corresponds to the RHS of (3.2.1) for symbol f with $\hat{\beta}$'s parameters and without the error term nor $\tilde{\Omega}(2nt)$, and

$$\begin{aligned} \Sigma'(t) = \Bigg[&\left(\frac{z_m - e^t}{z_m - e^{-t}}\right)^{\alpha_m+\hat{\beta}_m} \exp\left\{2\sum_{k=1}^{\infty} V_k(\sinh(tk))\right\}\left(\frac{2t}{1-e^{-2t}}\right)^{\alpha_0-\beta_0} \\ &\times (1-e^{-t}z_j)^{\sum_{j=1}^{m-1}(\alpha_j-\hat{\beta}_j)} \times (1-e^{-t}z_j^{-1})^{\sum_{j=1}^{m-1} -(\alpha_j+\hat{\beta}_j)} \\ &+ \left(\frac{z_m - e^t}{z_m - e^{-t}}\right)^{\alpha_m-\hat{\beta_m}} \exp\left\{-2\sum_{k=1}^{\infty} V_{-k}(\sinh(tk))\right\} \times \left(\frac{2t}{1-e^{-2t}}\right)^{-(\alpha_0+\beta_0)} \\ &\times (1-e^{-t}z_j)^{\sum_{j=1}^{m-1}(\alpha_j-\hat{\beta}_j)} \times (1-e^{-t}z_j^{-1})^{\sum_{j=1}^{m-1}(\alpha_j+\hat{\beta}_j)}\Bigg]. \end{aligned} \tag{3.6.20}$$

Now, we will assume that we have more than one maximum of $\Re\beta_j$ and that $l=2$. Then, by rewriting the real parts of β_j in an increasing order, we have

$$\Re\beta_0 < \Re\beta_1 \le ... \le \Re\beta_{m-2} < \Re\beta_{m-1} = \Re\beta_m \tag{3.6.21}$$

Define the function $\tilde{f}_t(z)$ by the following changes:

$\Re\tilde{\beta}_j = \Re\beta_j$ for $j=0,...,m-2$,

$\Re\tilde{\beta}_j = \Re\beta_j - 1$ for $j=m-1,m$,

and

$$f(z,t) = z^2 \prod_{j=m-1}^{m} z_j^{-1}\tilde{f}_t(z), \tag{3.6.22}$$

which can be dealt with using Lemma 3.6.3. Then we must evalute the determinants $F_n, n >$

N_0 for a sufficiently large $N_0 > 0$. By using (3.4.155) and (3.6.11), we have

$$\frac{\phi_{n+r}(0)}{\chi_{n+r}} = D(0)^{-1}\rho_{n+r}(0), \tag{3.6.23}$$
$$\rho_{n+r}(z) = -\sum_{k=1}^{m}\frac{A_{k,12}^{n+r}}{z-z_k} - \frac{A_{e^t,12}}{z-e^t} - \frac{A_{e^{-t},12}}{z-e^{-t}} + \mathcal{O}\big(\delta\max_k\frac{n^{-2\Re\beta_k}}{n}\big).$$

This expansion is uniform and differentiable at $z=0$ and

$$F_n = D(0)^{-2}\begin{vmatrix}\rho_n(0) & \rho_{n+1}(0)\\ \frac{d}{dz}\rho_n(0) & \frac{d}{dz}\rho_{n+1}(0)\end{vmatrix}. \tag{3.6.24}$$

Then we obtain the derivative of $\rho(z)$ in (3.6.24) as $|||\Re\tilde{\beta}_j - \Re\tilde{\beta}_k||| < 1$ and $\alpha_j \pm \tilde{\beta}_k \notin \mathbb{Z}_-$,

$$\rho_{n+r}(0) = \sum_{j=m-1}^{m} d_j z_j^{n+r} + \frac{A_{e^t,12}}{e^t} + \frac{A_{e^{-t},12}}{e^{-t}} + \mathcal{O}\big(\delta\max_k\frac{n^{-2\Re\beta_k}}{n}\big) \tag{3.6.25}$$

and

$$\frac{d^s}{dz^s}\rho_{n+r}(0) = s!\sum_{j=m-1}^{m} d_j z_j^{n+r-s} + \mathcal{O}([\delta+\frac{1}{n}]n^{-2\Re\tilde{\beta}_j-1}) \tag{3.6.26}$$

where

$$d_j = (1-z_je^{-t})^{(\alpha_0+\tilde{\beta}_0)} \times (1-e^{-t}z_j^{-1})^{-(\alpha_0-\tilde{\beta}_0)} \times e^{t(\alpha_0+\tilde{\beta}_0)}n^{-2\tilde{\beta}_j-1}$$
$$\times\mu_j\frac{\Gamma(1+\alpha_j+\tilde{\beta}_j)}{\Gamma(\alpha_j-\tilde{\beta}_j)}(1+\mathcal{O}(u)) \tag{3.6.27}$$

and

$$\mu_j = e^{V_0}\frac{\exp\Big\{\sum_{k=1}^{\infty}V_kz_j^k\Big\}}{\exp\Big\{-\sum_{k=1}^{\infty}V_{-k}z_j^{-k}\Big\}}\exp\Big\{-i\pi\Big(\sum_{k=1}^{j-1}\alpha_k - \sum_{k=j+1}^{m}\alpha_k\Big)\Big\}\prod_{k\neq j}\Big(\frac{z_j}{z_k}\Big)^{\alpha_k}|z_j-z_k|^{2\beta_k}. \tag{3.6.28}$$

When these expressions are substituted into the determinant F_n and we use (3.4.159), (3.4.160), we obtain

$$F_n = D(0)^{-2}\Bigg[d_{m-1}d_m z_{m-1}^n z_{m-1}^n |z_{m-1} - z_m|^2 + \left(\sum_{j=m-1}^{m} d_j z_j^n - \sum_{j=m-1}^{m} d_j z_j^{n-1}\right) \times \left(\frac{A_{e^t,12}}{e^t} + \frac{A_{e^{-t},12}}{e^{-t}}\right)\Bigg](1+o(1)). \tag{3.6.29}$$

Consequently, we have

$$D_n(z^2\tilde{f}_t(z)) = \prod_{j=m-1}^{m} z_j^{-n} D(0)^{-2}\Bigg[d_{m-1}d_m z_{m-1}^n z_m^n |z_{m-1} - z_m|^2 + \left(\sum_{j=m-1}^{m} d_j z_j^n - \sum_{j=m-1}^{m} d_j z_j^{n-1}\right) \times \left(\frac{A_{e^t,12}}{e^t} + \frac{A_{e^{-t},12}}{e^{-t}}\right)\Bigg] D_n(\tilde{f}_t(z))(1+o(1)) \tag{3.6.30}$$

Then, using the previous results Theorem 3.2.1 with seminorm less than one for $(D_n(f_t(z)))$

relative to $\tilde{\beta}$-parameters and substituting them in (3.6.30), we get

$$
\begin{aligned}
D_n(f_t(z)) &= D_n\Big(f(z;\alpha_0,\alpha_j,\alpha_{m-1},\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_{m-1}+1,\tilde{\beta}_m+1)\Big)\times\tilde{\Omega}(2nt)(1+o(1))\\
&+\Big(\prod_{j=m-1}^{m} z_j^{-n}\Big)\Big[\exp\Big\{nV_0+nt(\alpha_0+\tilde{\beta}_0)\Big\}\exp\Big\{\sum_{k=1}^{\infty}k\Big[V_k-(\alpha_0+\tilde{\beta}_0+1)\frac{e^{-kt}}{k}\Big]\\
&\times\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0)\frac{e^{-kt}}{k}\Big]\Big\}\times n^{\sum_{j=1}^{m}(\alpha_j^2-\tilde{\beta}_j{}^2)}\times n^{-2\tilde{\beta}_0-1}\times\exp\Big\{-\sum_{j=1}^{m-2}(\alpha_j-\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k\\
&-(\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^k\Big\}\times\exp\Big\{-\sum_{j=m-1}^{m}(\alpha_j-\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\tilde{\beta}_0+1)\frac{e^{-tk}}{k}\Big]z_j^k\Big\}\\
&\times\prod_{j=m-1}^{m}(1-e^tz_j^{-1})^{\alpha_j+\tilde{\beta}_j}\times\exp\Big\{-\sum_{j=1}^{m}(\alpha_j+\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\}\\
&\times\prod_{j=1}^{m}G_{\alpha_j+\tilde{\beta}_j,\alpha_j-\tilde{\beta}_j}\times\frac{G_{\alpha_0+\tilde{\beta}_0+1,\alpha_0-\tilde{\beta}_0-1}}{\Gamma(1+\alpha_0+\tilde{\beta}_0)}\times(2t)^{\alpha_0-\tilde{\beta}_0}\times\exp\Big\{\sum_{k=1}^{\infty}V_ke^{tk}\Big\}\\
&\times\prod_{1\le j\le k<m}|z_j-z_k|^{2\left(\beta_k\tilde{\beta}_j-\alpha_j\alpha_k\right)}\left(\frac{z_k}{z_je^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\tilde{\beta}_j\right)}\times k(2nt)\times\tilde{\Omega}(2nt)\\
&+\exp\Big\{nV_0+nt(\alpha_0+\tilde{\beta}_0)\Big\}\times\exp\Big\{\sum_{k=1}^{\infty}k\Big[V_k-(\alpha_0+\tilde{\beta}_0)\frac{e^{-kt}}{k}\Big]\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0-1)\frac{e^{-kt}}{k}\Big]\Big\}\\
&\times\exp\Big\{-\sum_{j=1}^{m}(\alpha_j-\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^k\Big\}\exp\Big\{-\sum_{j=1}^{m-2}(\alpha_j+\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k}\\
&-(\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\}\times\exp\Big\{-\sum_{j=m-1}^{m}(\alpha_j+\tilde{\beta}_j)\sum_{k=1}^{\infty}\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0-1)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\}\\
&\times\exp\Big\{-\sum_{k=1}^{\infty}V_{-k}e^{tk}\Big\}\times\prod_{j=1}^{m}G_{\alpha_j+\tilde{\beta}_j,\alpha_j-\tilde{\beta}_j}\times\frac{G_{\alpha_0+\tilde{\beta}_0+1,\alpha_0-\tilde{\beta}_0-1}}{\Gamma(1+\alpha_0+\tilde{\beta}_0)}\times n^{\sum_{j=1}^{m}(\alpha_j^2-\tilde{\beta}_j{}^2)}n^{-2\tilde{\beta}_0-1}\\
&\times(2t)^{-(\alpha_0+\tilde{\beta}_0)}\times\prod_{j=m-1}^{m}(1-e^tz_j)^{-(\alpha_j-\tilde{\beta}_j)}\times k(2nt)\times\prod_{1\le j\le k<m}|z_j-z_k|^{2\left(\beta_k\tilde{\beta}_j-\alpha_j\alpha_k\right)}\\
&\times\left(\frac{z_k}{z_je^{i\pi}}\right)^{\left(\alpha_j\beta_k-\alpha_k\tilde{\beta}_j\right)}\times\tilde{\Omega}(2nt)\Big]\times\sum_{j=m-1}^{m}(z_j^n-z_j^{n-1})\Big[\exp\Big\{\sum_{k=1}^{\infty}\Big[V_k-(\alpha_0+\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^k\Big\}\\
&\exp\Big\{\sum_{k=1}^{\infty}-\Big[V_{-k}-(\alpha_0-\tilde{\beta}_0)\frac{e^{-tk}}{k}\Big]z_j^{-k}\Big\}\times v_j\times\frac{\Gamma(1+\alpha_j+\tilde{\beta}_j)}{\Gamma(\alpha_j-\tilde{\beta}_j)}n^{-2\tilde{\beta}_j-1}\Big]\Big(1+o(1)\Big)
\end{aligned}
\tag{3.6.31}
$$

where,

$$v_j = \exp\Big\{ - i\pi\Big(\sum_{k=1}^{j-1}\alpha_k - \sum_{k=j+1}^{m}\alpha_k\Big)\Big\} \prod_{k\neq j}\Big(\frac{z_j}{z_k}\Big)^{\alpha_k}|z_j - z_k|^{2\tilde{\beta}_k}. \tag{3.6.32}$$

Then by simplifying the last equation, we have

$$\begin{aligned} D_n(f_t(z)) = {} & D_n\Big(f(z;\alpha_0,\alpha_j,\alpha_{m-1},\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_{m-1}+1,\tilde{\beta}_m+1)\Big) \times \tilde{\Omega}(2nt)(1+o(1)) \\ & + D_n\Big(f(z;\alpha_0,\alpha_j,\alpha_{m-1},\alpha_m,\tilde{\beta}_0,\tilde{\beta}_j,\tilde{\beta}_{m-1}+1,\tilde{\beta}_m+1)\Big) \times n^{-2\beta_0-1}K(2nt)\tilde{\Omega}(2nt) \\ & \times \frac{1}{\Gamma(\alpha_0-\beta_0)} \prod_{j=m-1}^{m}\Big(z_j^{-n}\Big)^n \sum_{j=m-1}^{m}(z_j^n - z_j^{n-1})\Sigma(t)(1+o(1)) \end{aligned} \tag{3.6.33}$$

where

$$\begin{aligned} \Sigma(t) = {} & \Big(\frac{2t}{1-e^{-2t}}\Big)^{\alpha_0-\beta_0} \prod_{j=m-1}^{m}(1-e^t z_j^{-1})^{\alpha_j+\tilde{\beta}_j}(1-e^{-t}z_j)^{-(\alpha_j-\tilde{\beta}_j)} \exp\Big\{-\sum_{k=1}^{\infty}e^{-tk}V_{-k}\Big\} \\ & \times \exp\Big\{\sum_{k=1}^{\infty}e^{-tk}V_k\Big\} + \Big(\frac{1-e^{-2t}}{2t}\Big)^{\alpha_0+\beta_0} \prod_{j=m-1}^{m}(1-e^t z_j)^{-(\alpha_j-\tilde{\beta}_j)}(1-e^{-t}z_j^{-1})^{\alpha_j+\tilde{\beta}_j} \\ & \times \exp\Big\{-\sum_{k=1}^{\infty}e^{-tk}V_k\Big\}\exp\Big\{-\sum_{k=1}^{\infty}e^{tk}V_{-k}\Big\}. \end{aligned} \tag{3.6.34}$$

3.6.2 The second case

Suppose that $|||\beta_1, ..., \beta_m||| = 1$ for $t > 0$ and at the emerging singularity $|||\beta_0, ..., \beta_m||| = 1$. This will be divided to two possible subcases. Case 1: $\Re\beta_0 = \min\{\Re\beta_j : 1 \le j \le m\}$ or $\Re\beta_0 = \max\{\Re\beta_j : 1 \le j \le m\}$, and Case 2: $\min\Re\beta_j < \Re\beta_0 < \max\Re\beta_j$.

Let us assume without loss of generality , and by relabeling β_j according to the increasing real part, we get

$$\Re\beta_0 = \Re\beta_1 = ... = \Re\beta_p < \Re\beta_{p+1} = ... = \Re\beta_{m-l} < \Re\beta_{m-l+1} = ... = \Re\beta_m$$

Then, using the same procedure as in the first case where $|||\beta_1, ..., \beta_m||| < 1$ for $t > 0$ and at the emerging singularity $|||\beta_0, ..., \beta_m||| = 1.$, we will define $\tilde{f}_t(z)$ with $|||\tilde{\beta}^{(t)}||| < 1$, and we will compute the asymptotics for the original $f_t(z)$ using the asymptotic behaviour of Toeplitz determinants with respect to $\tilde{f}_t(z)$ and

$$f_t(z) = (-1)^l \prod_{j=1}^{l} z_j^{-1} z^l \tilde{f}_t(z). \tag{3.6.35}$$

Then, by using (3.6.35) and Lemma 3.6.3, we get

$$D_n(f_t(z)) = (-1)^{n(l+1)} \prod_{j=1}^{l} z_j^{-n} \frac{F_n}{\prod_{j=1}^{l-1} j!} D_n(\tilde{f}_t(z)). \tag{3.6.36}$$

For example, if we assume that $l = 2$ this gives us the same result as in (3.6.33).

For the case when $\min \Re\beta_j < \Re\beta_0 < \max \Re\beta_j$, the above gives us the result proved in Theorem 1.13 of [16].

3.7 Open problems

1. In [5], the authors computed the asymptotic expansion for Toeplitz determinants with matrix-valued symbols that possess jump singularities β_j when $|||\beta||| < 1$. However, it is still an open problem to describe the asymptotics for Toeplitz determinants in the block case when $|||\beta||| = 1$. In addition, in the block case, there is no description of the asymptotics of Toeplitz determinants $D_n(f)$ when the symbol possesses root type singularities $\alpha_j \neq 0$.

2. It would also be interesting to compute the double-scaling limits of Toeplitz determinants with scalar symbols using operator theoretic methods, where the main difficulty is related to the connection with the Painlevé V equations (see [16] for how the connection can be obtained using the Riemann-Hilbert approach).

3. Finally, notice that so far there is no description of double-scaling limits of Toeplitz determinants with matrix-valued symbols using operator theoretic or Riemann-Hilbert methods.

Chapter 4

Ising correlations above the critical temperature

The characterization of the diagonal and horizontal two-point correlation functions in the 2D Ising model using the Toeplitz determinant and other means is considered to be one of the most remarkable and groundbreaking results in statistical mechanics. In the physics and mathematics literature, it is proven that at $T = T_c$ the correlation functions decay like $n^{-1/4}$ and the double scaling limit as $n \to \infty$ and $T \nearrow T_c$ is described by a special Painlevé V transcendent. In the high-temperature regime, it is known that the correlations decay exponentially fast in physics and it has been described briefly in [17]. In this chapter, we will extend the description of this behaviour by using the Riemann-Hilbert approach for the diagonal directions.

4.1 Introduction

In this chapter, our analysis is based on the Riemann-Hilbert problem. We will connect Toeplitz determinants to orthogonal polynomials $\phi_n(z)$ and $\hat{\phi}_n(z^{-1})$ which satisfy the conditions in (2.5.1) with respect to the symbol $\eta(z)$ on the unit circle, and is given by

$$\eta(z;t) = e^{i\pi/2}(z-k)^{-1/2}(z-k^{-1})^{1/2}z^{1/2} \tag{4.1.1}$$

Then we find the asymptotics of Toeplitz determinants by analyzing the corresponding Riemann-Hilbert problem.

4.1.1 Riemann-Hilbert problem for OPUC with Szegő type-symbols

Recalling the matrix-valued function (3.3.1), as in Chapter 3, $Y(z)$ is the unique solution to the following Riemann-Hilbert problem with respect to the Szegő-type symbol.

RH-Y1: $Y : \mathbb{C} \setminus \mathbb{T} \to \mathbb{C}^{2\times 2}$ is analytic,

RH-Y2: Let $z \in \mathbb{T} \setminus \cup_{j=1}^{m} z_j$, where $j = 1, \cdots, m$, $Y(z)$ has continuous boundary values $Y_+(z)$ and $Y_-(z)$, related by the jump condition

$$Y_+(z) = Y_-(z) \begin{pmatrix} 1 & z^{-n}\eta(z) \\ 0 & 1 \end{pmatrix}, \quad z \in \mathbb{T}, \tag{4.1.2}$$

RH-Y3: It has the following asymptotic behaviour as $z \to \infty$:

$$Y(z) = (I + \mathcal{O}(\frac{1}{z})) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \tag{4.1.3}$$

4.1.2 Solution for Riemann-Hilbert problem $Y(z)$

Here, again we solve the Riemann-Hilbert problem for $Y(z)$ using the steepest descent technique [21]. The standard steepest descent has been introduced in the Appendices of [4]. To begin, we define $T(z)$ to normalize the function at $z \to \infty$ as follows:

$$T(z) = Y(z) \begin{cases} z^{-n\sigma_3} & : |z| > 1 \\ I & : |z| < 1 \end{cases}. \tag{4.1.4}$$

This solves the following Riemann-Hilbert problem:

RH-T1: It is analytic for $z \in \mathbb{C} \setminus \mathbb{T}$,

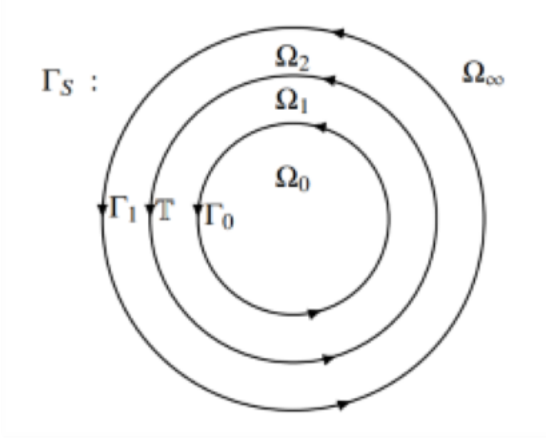

Figure 4.1: R-H problem for $S(z)$

RH-T2: The jump condition on $z \in \mathbb{T}$ is

$$T_+(z) = T_-(z) \begin{pmatrix} z^n & \eta(z) \\ 0 & z^{-n} \end{pmatrix}, \tag{4.1.5}$$

RH-T3: As $z \to \infty$, the function has the following behaviour:

$$T(z) = \big(I + \mathcal{O}(z^{-1})\big).$$

The next step will deform the unit circle to solve the oscillations in the jump matrix (4.1.5) when $n \to \infty$. Let

$$S(z) = \begin{cases} T(z), & \text{for } z \in \Omega_0 \cup \Omega_\infty, \\ T(z) \begin{pmatrix} 1 & 0 \\ \eta(z)^{-1} z^{-n} & 1 \end{pmatrix}, & \text{for } z \in \Omega_2, \\ T(z) \begin{pmatrix} 1 & 0 \\ -\eta(z)^{-1} z^{n} & 1 \end{pmatrix}, & \text{for } z \in \Omega_1 \end{cases}, \tag{4.1.6}$$

where we factorize the jump matrix as follows:

$$\begin{pmatrix} z^n & \eta(z) \\ 0 & z^{-n} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ z^{-n}\eta(z)^{-1} & 1 \end{pmatrix} \begin{pmatrix} 0 & \eta(z) \\ -\eta(z)^{-1} & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ z^{n}\eta(z)^{-1} & 1 \end{pmatrix} \tag{4.1.7}$$
$$= J_1(z,n)J^{(\infty)}J_2(z,n).$$

Observe that the matrices J_1 and J_2 tend to the identity matrix uniformly on their respective contours, and exponentially quickly as $n \to \infty$. The function $S(z)$ can be used to solve the following Riemann-Hilbert problem:

RH-S1: $S(z)$ is analytic for $z \in \mathbb{C} \setminus \Gamma_s$, where $\Gamma_s = \Gamma_0 \cup \mathbb{T} \cup \Gamma_1$,

RH-S2: The boundary values are defined by the following jump conditions:

$$S_+(z) = S_-(z)J_s(z,n), \quad z \in \Gamma_s,$$

where

$$J_s(z,n) = \begin{cases} \begin{pmatrix} 1 & 0 \\ z^{-n}\eta(z)^{-1} & 1 \end{pmatrix}, & z \in \Gamma_0 \\ \begin{pmatrix} 0 & \eta(z) \\ -\eta(z)^{-1} & 0 \end{pmatrix}, & z \in \mathbb{T} \\ \begin{pmatrix} 1 & 0 \\ z^{n}\eta(z)^{-1} & 1 \end{pmatrix}, & z \in \Gamma_1 \end{cases}, \tag{4.1.8}$$

RH-S3: It has the following asymptotic behaviour as $z \to \infty$:

$$S(z) = \big(I + \mathcal{O}(z^{-1})\big).$$

4.1.3 Global parametrix $N(z)$

We assume the following Riemann-Hilbert problem for $N(z)$.

RH-N1: It is analytic for $z \in \mathbb{C} \setminus \mathbb{T}$,

RH-N2: The boundary values are determined by the following jump conditions:

$$N_+(z) = N_-(z) \begin{pmatrix} 0 & \eta(z) \\ -\eta^{-1} & 0 \end{pmatrix}, \quad \text{for } z \in \mathbb{T}, \tag{4.1.9}$$

RH-N3: As $z \to \infty$, the function has the following behaviour:

$$N(z) = \big(I + \mathcal{O}(z^{-1})\big). \tag{4.1.10}$$

The above Riemann-Hilbert problem for $N(z)$ has the following unique solution:

$$N(z) = \begin{cases} D(z)^{\sigma_3} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, & \text{for } \quad |z| < 1 \\ D(z)^{\sigma_3}, & \text{for } \quad |z| > 1 \end{cases}, \tag{4.1.11}$$

where $D(z)$ is the unique function related to $\eta(z)$, it is analytic in $\mathbb{C} \setminus \mathbb{T}$ with the following jump condition:

$$D_+(z) D_-^{-1}(z) = \eta(z), \quad z \in \mathbb{T}. \tag{4.1.12}$$

By the Plemelj-Sokhotski formulas, we have

$$D(z) = \exp\left[\frac{1}{2\pi i} \int_{\mathbb{T}} \frac{\log(\eta(s)) ds}{s - z}\right]. \tag{4.1.13}$$

The function $D(z)$ (Szegő function) has the following behaviour as $z \to \infty$:

$$D(z) = 1 + \mathcal{O}(1/z).$$

4.1.4 Small Riemann-Hilbert problem

Let us introduce the function

$$R(z,n) = S(z,n)N^{-1}(z,n). \tag{4.1.14}$$

The Riemann-Hilbert problem for $R(z)$ is given as follows:

RH-R1 $R(z)$ is analytic in $\mathbb{C} \setminus (\Gamma_1 \cup \Gamma_0)$,

RH-R2 $R_+(z,n) = R_-(z,n)J_R(z,n), \quad z \in \Gamma_0 \cup \Gamma_1 = \Sigma_R$,

RH-R3 $R(z,n) = I + \mathcal{O}(1/z)$ as $z \to \infty$.

For large n, this Riemann-Hilbert problem is solvable and it can be written as follows:

$$R(z,n) = I + R_1(z,n) + R_2(z,n) + \cdots, \tag{4.1.15}$$

where for $k \geq 1$

$$R_k(z,n) = \frac{1}{2\pi i}\int_{\mathbb{T}} \frac{[R_{k-1}(\tau;n)]_-(J_R(\tau;n)-I)}{\tau - z}d\tau, \quad z \in \mathbb{C} \setminus \Sigma_R. \tag{4.1.16}$$

In order to calculate $R_1(z,n)$, we have

$$J_R - I = \begin{cases} \begin{pmatrix} 0 & z^n\eta^{-1}(z)\beta^2(z) \\ 0 & 0 \end{pmatrix}, & z \in \Gamma_0, \\ \begin{pmatrix} 0 & 0 \\ z^{-n}\eta^{-1}(z)\beta^{-2}(z) & 0 \end{pmatrix}, & z \in \Gamma_1. \end{cases} \tag{4.1.17}$$

Thus, we obtain

$$R_1(z,n) = \begin{pmatrix} 0 & \frac{-1}{2\pi i}\int_{\Gamma_0} \frac{\tau^n\eta^{-1}(\tau)\beta^2(\tau)}{\tau - z}d\tau \\ \frac{1}{2\pi i}\int_{\Gamma_1} \frac{\tau^{-n}\eta^{-1}(\tau)\beta^{-2}(\tau)}{\tau - z}d\tau & 0 \end{pmatrix}. \tag{4.1.18}$$

The Riemann-Hilbert problems for $Y(z)$ can be solved by going back to the steps $R \to S \to T \to Y$. Indeed,

$$Y(z,n) = R(z,n) \begin{cases} \begin{pmatrix} D(z) & 0 \\ 0 & D^{-1}(z) \end{pmatrix} z^{n\sigma_3}, & z \in \Omega_\infty \\ \begin{pmatrix} D(z) & 0 \\ -z^{-n}D^{-1}(z)\eta^{-1}(z) & D^{-1}(z) \end{pmatrix} z^{n\sigma_3}, & z \in \Omega_2 \\ \begin{pmatrix} z^n D(z)\eta^{-1}(z) & D(z) \\ -D^{-1}(z) & 0 \end{pmatrix}, & z \in \Omega_1 \\ \begin{pmatrix} 0 & D(z) \\ -D^{-1}(z) & 0 \end{pmatrix} z^{n\sigma_3}, & z \in \Omega_0 \end{cases}, \tag{4.1.19}$$

where for $z \in \mathbb{C} \setminus \Sigma_R$, we have

$$R(z,n) = \begin{pmatrix} 1 + \mathcal{O}(\frac{\rho^{-2n}}{1+|z|}) & R_{1,12}(z,n) + \mathcal{O}(\frac{\rho^{-3n}}{1+|z|}) \\ R_{1,21} + \mathcal{O}(\frac{\rho^{-3n}}{1+|z|}) & 1 + \mathcal{O}(\frac{\rho^{-2n}}{1+|z|}) \end{pmatrix}, \qquad \text{as } n \to \infty. \tag{4.1.20}$$

4.2 Ising models and Toeplitz determinants

One of the most important models in statistical mechanics is the $2D$ Ising model (See [17]), which was solved by Onsager in [41]. It is a $2M \times 2N$ rectangular lattice of $\mathbb{Z}^2$ that involves the interactions of random spins σ_i and σ_j taking values 1 and -1 at each site (i,j), where $-M \leq i \leq, M-1$, and $-N \leq j \leq N-1$. The number of possible spin configurations of the lattice related to values of σ_{ij} is 2^{MN}. The nearest neighbour spins interactions are the most interesting, and the total interaction energy is given by

$$E(\sigma) = -\sum_{j=-M}^{M-1} \sum_{i=-N}^{N-1} \big(J_h \sigma_{ji}\sigma_{ji+1} + J_v \sigma_{ji}\sigma_{j+1i}\big), \quad J_h, J_v > 0, \tag{4.2.1}$$

where J_h, J_v are the horizontal and vertical interaction as the nearest constants What do you mean by this sentence?. The system is called ferromagnetic, i.e., parallel spins, if J_h and J_v

are positive. It has lower energy than the anti-parallel spins, where J_h and J_v are negative. The associated normalized Gibbs measure is given by

$$Pr(\sigma) = \frac{1}{Z(T)} e^{-E\sigma/K_B T},$$

where $K_B T$ is the Boltzmann's constant and T is the temperature. Note that the partition function is given by

$$Z(T) = \sum_{\sigma} e^{-E(\sigma)/k_B T}, \tag{4.2.2}$$

where the sum is being taken over all possible configurations. The existence of a thermodynamic phase transition is the most significant aspect of the 2D Ising model as the size of the lattice becomes infinitely large at a certain critical temperature T_c which is dependent on J_h and J_v.

The two-spin correlation function is defined by the following expression:

$$< \sigma_{0,0}\sigma_{N,M} > = \lim_{M,N\to\infty} \frac{1}{Z(T)} \sum_{\sigma} \sigma_{0,0}\sigma_{N,M} e^{-E(\sigma)/T}. \tag{4.2.3}$$

Using this function, we can evaluate the magnetization by measuring the long-range order in the lattice at temperature T. As an example, a bar magnet has a critical temperature T_c, also known as the Curie point. Below $T < T_c$, it spontaneously exhibits magnetization, and above $T > T_c$, it does not (in the absence of an external field). The one-dimentional Ising model fails to go through a phase transition at any temperature as demonstrated by Ising [27]. However, in two or three dimensions, it does indeed exhibit spontaneous magnetization, as shown by Peierls in [45] whose work included an incorrect step which was corrected by Griffiths [26] many years later. Kramers and Wannier obtained the first exact quantitative result for the $2D$ Ising model in 1941 [35], when they formulated the following formula for T_c. In the situation where $J_h = J_v = J$,

$$\sinh\left(\frac{2J}{T_c}\right) = 1. \tag{4.2.4}$$

For the square lattice of size $N \times N$, Onsager formulated the partition function [44]. The analysis done by Onsager makes possible a variety of interaction constants $J_h \neq J_v$ what does it mean?

$$\sinh\frac{2J_h}{K_BT_c}\sinh\frac{2J_v}{K_BT_c} = k = e^t = 1. \tag{4.2.5}$$

4.2.1 The horizontal correlation functions $< \sigma_{1,1}\sigma_{1,1+n} >$

In 1949, Onsager and Kaufman presented their result for the spontaneous magnetization M of the two-dimensional Ising model [32] as follows:

$$M = (1-k_{ons}^2)^{1/8}, \quad k_{ons} = \left(\sinh\frac{2J_h}{K_BT_c}\sinh\frac{2J_v}{K_BT_c}\right)^{-1}, \tag{4.2.6}$$

where $0 < k_{ons} < 1 \Longleftrightarrow T < T_c$, and $T > T_c, \Longleftrightarrow k_{ons} > 1$? and in this case $M = 0$ what case?. By some physical arguments, the spontaneous magnetization M was shown to be given by

$$M = \sqrt{\lim_{n\to\infty} < \sigma_{1,1}\sigma_{1,1+n} >}. \tag{4.2.7}$$

In [32], the expression $< \sigma_{1,1}\sigma_{1,1+n} >$ was given by Kaufman and Onsager as a sum of two Toeplitz determinants. Therefore, the challenge that Kaufman and Onsager faced to compute M through the formula (4.2.7), was to compute the asymptotics of $n \times n$ Toeplitz determinants as n approaches infinity. Nevertheless, at that time, the only result that was known to exist was szegő's result with an unknown error term $o(n)$. However, in order to compute the magnetization M, it is essential to describe the error terms. Three years passed between Onsager's 1948 and 1949 announcements of formula (4.2.6) for M without proof, and Yang's successful discovery of the formula's derivation in 1952 [50]. Yang's methodology is based on the findings of Kaufman and Onsager, and it does not employ Toeplitz determinants directly. He proved the result in the case $J_h = J_v$, with $T < T_c$, which was then proved in the general case where $J_h \neq J_v$ by Chang [12].

In 1955, Potts and Ward [PotWar], demonstrated that the correlation function $< \sigma_{1,1}\sigma_{1,1+n} >$ along a row for the two-dimensional Ising model could be represented using a single Toeplitz determinant, as opposed to the summation of two Toeplitz determinants, as follows:

$$< \sigma_{1,1}\sigma_{1,1+n} >= D_n(\phi_{ons}). \tag{4.2.8}$$

The function ϕ_{ons} is the Onsager function

$$\phi_{ons}(e^{i\theta}) = \left[\left(\frac{1-\gamma_1 e^{i\theta}}{1-\gamma_1 e^{-i\theta}}\right)\left(\frac{1-\gamma_2 e^{-i\theta}}{1-\gamma_1 e^{i\theta}}\right)\right]^{1/2}, \tag{4.2.9}$$

where $\gamma_1 = z_1 z_2^*$, $\gamma_2 = z_2^*/z_1$, $z_1 = \tanh\frac{J_h}{K_B T}$, $z_2 = \tanh\frac{J_v}{K_B T}$, and $z_2^* = \frac{1-z_2}{1+z_2}$. Notice that

$$\gamma_2 \lesseqgtr 1 \Longleftrightarrow k_{ons} \lesseqgtr 1. \tag{4.2.10}$$

To see the equivalence, see [17]. This implies that $T \lesseqgtr T_c$ corresponds to $z_2^* \lesseqgtr z_1$, respectively, and $T = T_c$ is equivalent to $z_2^* = z_1$.

1. In [43], it was observed that when $T < T_c$, certain smoothness conditions are satisfied, $0 < \gamma_1$, and $\gamma_2 < 1$ by (4.2.10), and ϕ_{ons} does not possess winding on the unit circle. To determine the asymptotic behaviour of the correlation function, Theorem 2.3.3 is applied. Then in [49], Wu derived the higher-order terms as $n \to \infty$:

$$< \sigma_{1,1}, \sigma_{1,1+n} > = (1-k_{ons}^2)^{1/4}\left(1 + (2\pi n^2)^{-1}\gamma_2^{2n}(\gamma_2^{-1}-\gamma_2)^{-2}[1 + \frac{c_1}{n} + \frac{c_2}{n^2} + \cdots]\right), \tag{4.2.11}$$

where c_1, c_2 are given in terms of γ_1 and γ_2. If $\gamma_2 < 1$, $< \sigma_{1,1}, \sigma_{1,1+n} >\to M = (1-k_{ons}^2)^{1/4}$ exponentially fast.

2. For $T > T_c$, it is observed that $0 < \gamma_1 < 1 < \gamma_2$. Consequently, the winding number of ϕ_{ons} is determined to be -1. The asymptotic behaviour of ϕ_{ons} has been described in the study conducted by Wu in [49], as follows:

$$\begin{aligned}< \sigma_{1,1}, \sigma_{1,1+n} > = {} & (\pi n)^{-1/2}\gamma_2^{-n}(1-\gamma_1^2)^{1/4}(1-\gamma_2^{-2})^{-1/4}(1-\gamma_1\gamma_2)^{-1/2} \\ & \times (1 + \frac{A_1}{n} + \frac{A_2}{n^2} + \cdots),\end{aligned} \tag{4.2.12}$$

with explicit expressions for A_1, A_2 in terms of γ_1 and γ_2 .

3. Wu further demonstrated that for $T = T_c$, the function ϕ_{ons} has a jump discontinuity at $\theta = 0$ and is asymptotically represented by

$$< \sigma_{1,1}, \sigma_{1,1+n} > = e^{1/4} 2^{1/12} A^{-3} n^{-1/4} (1+\gamma_1)^{1/4} (1-\gamma_1)^{-1/4} \times (1 + B_1 n^{-2} + \mathcal{O}(n^{-3})), \quad (4.2.13)$$

where A is Glaisher's constant

$$A = e^{1/12} e^{-\xi'(-1)}.$$

Here ξ is the Riemann's zeta function and B_1 is expressed explicitly in terms of γ_1.

4.2.2 The diagonal correlation function $< \sigma_{0,0}\sigma_{n,n} >$

It is a remarkable fact that the two-spin correlation function is a Toeplitz determinant

$$< \sigma_{0,0}\sigma_{n,n} > = e^{nt/2} D_n(f(z,t)), \quad (4.2.14)$$

where

$$f(z,t) = e^{V(z)} z^{\sum_{j=0}^{m} \beta_j} \prod_{j=0}^{m} |z - z_j|^{2\alpha_j} g_{z_j,\beta_j}(z) z_j^{-\beta_j}, \quad \theta \in [0, 2\pi). \quad (4.2.15)$$

This symbol has the following properties depending on whether temperature T is below, equal to, or above the critical temperature T_c:

1. For $T < T_c \Longleftrightarrow t > 0$, using the symbol (4.2.15), there is no Fisher-Hartwig singularity at $z = 1$. In this case, the determinant can be given by the strong Szegő theorem, because the function is analytic in a neighbourhood of the unit circle. In [42], the diagonal long-range order in low-temperature regime has been given by:

$$M = \sqrt{\lim_{n\to\infty} < \sigma_{00}\sigma_{nn} >} = (1 - k^{-2})^{1/8}. \quad (4.2.16)$$

2. For $T = T_c \iff t = 0$, there is a Fisher-Hartwig singularity at $z = 1$ with parameter $\alpha = 0$, and $\beta = -\frac{1}{2}$. Using [41] and [40] , we obtain the following asymptotics

$$D_n(0) = \frac{(\pi)^{1/2} G(1/2)^2}{n^{1/4}} (1 + o(1)). \tag{4.2.17}$$

3. For the high-temperature regime $T > T_c \iff t < 0$, the symbol $f(z,t)$ has a Fisher-Hartwig singularity at $z = 1$ with $\alpha = 0$ and $\beta = -1$.

$$f(z,t) = z^{-1}\hat{\eta}(z), \tag{4.2.18}$$

where $\hat{\eta}(z)$ is the function associated to $2D$ Ising model and is given by

$$\hat{\eta}(z;t) = k^{1/2}\eta(z) = e^{i\pi/2}k^{1/2}(z-k)^{-1/2}(z-k^{-1})^{1/2}z^{1/2}. \tag{4.2.19}$$

This expression can be rewritten as follows:

$$\hat{\eta} = \sqrt{\frac{1-kz}{1-kz^{-1}}}. \tag{4.2.20}$$

To determine whether $f(z;t)$ is a Fisher-Hartwig symbol, we will show that $\arg f(1+0i,t) = \pi = -\arg f(1-0i,t)$. Indeed,

$$\begin{aligned} f(z;t) &= \exp\left\{\frac{1}{2}\log|z| + \frac{1}{2}\log|z-k^{-1}| - \frac{1}{2}\log|z-k|\right\} \\ &\times \exp i\left\{\frac{\pi}{2} - \frac{1}{2}(0) + \frac{\pi}{2} - \frac{1}{2}(0)\right\} \\ &= \exp\left\{\frac{1}{2}\log|z| + \frac{1}{2}\log|z-k^{-1}| - \frac{1}{2}\log|z-k|\right\}e^{i\pi}, \end{aligned} \tag{4.2.21}$$

which gives us,

$$\arg f(1+0i,t) = \pi, \quad \text{and} \quad \arg f(1-0i,t) = -\pi. \tag{4.2.22}$$

For the high-temperature regime $t < 0$, $\hat{\eta}(z)$ is analytic in $\mathbb{C} \setminus ([0,e^t] \cup [e^{-t},\infty])$ and the winding number is equal to zero. Therefore, $\hat{\eta}(z)$ is the Szegő symbol. To show this, it is sufficient to show that $\arg\hat{\eta}(1-0i,t) = \arg\hat{\eta}(1+0i,t)$. Note that

$$\hat{\eta}(z) = \exp\left\{\frac{1}{2}\log|z| - \frac{1}{2}\log|z-k| + \frac{1}{2}\log|z-k^{-1}|\right\}\exp i\Big\{\frac{\pi}{2} + \frac{1}{2}\arg z + \frac{1}{2}\arg(z-e^{t}) \tag{4.2.23}$$

$$+\frac{1}{2}\arg(z-e^{-t})\Big\}.$$

Then

$$\arg\hat{\eta}(1+0i,t) = \left\{\frac{\pi}{2} + \frac{1}{2}(0) - \frac{1}{2}(0) + \frac{1}{2}(\pi)\right\} = \pi, \tag{4.2.24}$$

and

$$\arg\hat{\eta}(1-0i,t) = \left\{\frac{\pi}{2} + \frac{1}{2}(2\pi) - \frac{1}{2}(2\pi) + \frac{1}{2}\pi\right\} = \pi. \tag{4.2.25}$$

For the symbol $f(z;t)$ in (4.2.18), the result in (2.4.11) does not hold because the Barnes G function will vanish if $\alpha \pm \beta \in \mathbb{Z}_-$.

In the high-temperature regime with $k < 1$, the asymptotic behaviour of the Toeplitz determinant was described in [17] as $n \to \infty$. Here will give more details:

To find the determinant, we will apply Lemma 3.6.3 to the function $f(z)$ in (4.2.18), which will give us the following:

$$D_n(f) = D_n(z^{-1}\hat{\eta}(z)) = (-1)^n\frac{\hat{\phi}(0)}{\chi_n}D_n(\hat{\eta}(z)). \tag{4.2.26}$$

We can find a piecewise analytic function $D(z)$, which solves the following scalar multiplicative Riemann-Hilbert problem:

$$D_+(z)D_-^{-1}(z) = \hat{\eta}(z), \quad z \in \mathbb{T}. \tag{4.2.27}$$

Thus, by using the Plemelj-Sokhotski formula we get

$$D(z) = \begin{cases} \sqrt{1-kz} & : |z| < 1 \\ \sqrt{1-kz^{-1}} & : |z| > 1 \end{cases}. \tag{4.2.28}$$

After that, to determine the asymptotics of the Toeplitz determinant, we need to compute the Fourier coefficients for the function $\hat{\eta}(z)$.

Firstly , we will compute $(\log \hat{\eta})_0$:

$$\begin{aligned}(\log\hat{\eta})_0 &= \frac{1}{2\pi i}\int \log\left((z-e^t)^{-1/2}(z-e^{-t})^{1/2}z^{1/2}e^{i\pi/2}e^{t/2}\right)\frac{dz}{z} \\ &= \frac{-1}{4i\pi}\int \log(z-e^t)\frac{dz}{z} + \frac{1}{4i\pi}\int\log(z-e^{-t})\frac{dz}{z} + \frac{1}{4i\pi}\int \log(z)\frac{dz}{z} \\ &+ \frac{1}{4i\pi}\int \log(e^{i\pi})\frac{dz}{z} + \frac{1}{4i\pi}\int\log(e^t)\frac{dz}{z}.\end{aligned} \tag{4.2.29}$$

Each term is calculated independently. Using the following expansion, $\log(z-e^t) = \log(z) - \sum_{j=1}^{\infty}\frac{e^{tj}z^{-j}}{j}$, and we have:

$$\begin{aligned}\frac{1}{4i\pi}\int\log(z-e^t)\frac{dz}{z} &= -\frac{1}{4i\pi}\int\log(z)\frac{dz}{z} + \frac{1}{4i\pi}\int\log(1-z^{-1}e^t)\frac{dz}{z} \\ &= -\frac{1}{4i\pi}\int\log(z)\frac{dz}{z}.\end{aligned} \tag{4.2.30}$$

Similarly, $\frac{1}{4i\pi}\int\log(z-e^{-t})\frac{dz}{z}$ can be calculated by using the expansion $\log(z-e^{-t}) = \log(-e^{-t}) - \sum_{j=1}^{\infty}\frac{e^{tj}z^j}{j}$:

$$\begin{aligned}\frac{1}{4i\pi}\int\log(z-e^{-t})\frac{dz}{z} &= \frac{1}{4i\pi}\int\log(e^{i\pi})\frac{dz}{z} - \frac{1}{4i\pi}\int\log(e^t)\frac{dz}{z} + \frac{1}{4i\pi}\int\log(1-ze^t)\frac{dz}{z} \\ &= \frac{1}{4i\pi}\int\log(e^{i\pi})\frac{dz}{z} - \frac{1}{4i\pi}\int\log(e^t)\frac{dz}{z} - \frac{1}{4i\pi}\int\sum_{j=1}^{\infty}\frac{e^{tj}z^j}{j}\frac{dz}{z} \\ &= \frac{1}{4i\pi}\int\log(e^{i\pi})\frac{dz}{z} - \frac{1}{4i\pi}\int\log(e^t)\frac{dz}{z}.\end{aligned} \tag{4.2.31}$$

Using (4.2.30) and (4.2.31), we obtain the following:

$$(\log\hat{\eta})_0 = i\pi. \tag{4.2.32}$$

In order to figure out $E(\hat{\eta})$, we will first find $(\log(\hat{\eta}))_k$ and $(\log(\hat{\eta}))_{-k}$, respectively.

$$\begin{aligned}
(\log\hat{\eta})_k &= \frac{1}{2\pi i}\int \log\Big(z^{1/2}(z-e^t)^{-1/2}(z-e^{-t})^{1/2}e^{i\pi/2}e^{t/2}\Big)z^{-k}\frac{dz}{z} \\
&= \frac{1}{4i\pi}\int(\log z)z^{-k}\frac{dz}{z} - \frac{1}{4i\pi}\int\log(z-e^t)z^{-k}\frac{dz}{z} + \frac{1}{4i\pi}\int\log(z-e^{-t})z^{-k}\frac{dz}{z} \\
&+ \frac{1}{4i\pi}\int\log e^{i\pi}z^{-k}\frac{dz}{z} + \frac{1}{4i\pi}\int\log e^t z^{-k}\frac{dz}{z}.
\end{aligned} \tag{4.2.33}$$

After that, we determine the integration for each term.

$$\begin{aligned}
\frac{-1}{4i\pi}\int\log(z-e^t)z^{-k}\frac{dz}{z}d &= \frac{-1}{4\pi}\int\log\Big(z(1-e^tz^{-1})\Big)z^{-k}\frac{dz}{z} \\
&= \frac{-1}{4i\pi}\int(\log z)z^{-k}\frac{dz}{z} - \frac{1}{4i\pi}\int\log\Big(1-e^tz^{-1}\Big)z^{-k}\frac{dz}{z} \\
&= \frac{-1}{4i\pi}\int(\log z)z^{-k}\frac{dz}{z} + \frac{1}{4i\pi}\int\sum_{j=1}^{\infty}\frac{e^{tj}z^{-j}}{j}z^{-k}\frac{dz}{z} \\
&= \frac{-1}{4i\pi}\int(\log z)z^{-k}\frac{dz}{z},
\end{aligned} \tag{4.2.34}$$

and we have

$$\begin{aligned}
\frac{1}{4i\pi}\int\log(z-e^{-t})z^{-k}\frac{dz}{z} &= \frac{1}{4i\pi}\int\log\Big(-e^{-t}(1-e^tz)\Big)z^{-k}\frac{dz}{z} \\
&= \frac{1}{4i\pi}\int\log e^{i\pi}z^{-k}\frac{dz}{z} + \frac{1}{4i\pi}\int\log(e^t)z^{-k}\frac{dz}{z} \\
&+ \frac{1}{4i\pi}\int\log\Big(1-e^tz\Big)z^{-k}\frac{dz}{z} \\
&= \frac{1}{4i\pi}\int\log e^{i\pi}z^{-k}\frac{dz}{z} + \frac{1}{4i\pi}\int\log(e^t)z^{-k}\frac{dz}{z} \\
&- \frac{1}{4i\pi}\int\sum_{j=1}^{\infty}\frac{e^{tj}z^j}{j}z^{-k}\frac{dz}{z} = -\frac{1}{2}\frac{e^{tk}}{k}.
\end{aligned} \tag{4.2.35}$$

Therefore, applying (4.2.34) and (4.2.35), we get the following:

$$(\log\hat{\eta})_k = \frac{-1}{2}\frac{e^{tk}}{k}. \tag{4.2.36}$$

In a similar manner, for $(\log\hat{\eta})_{-k}$ we get:

$$\begin{aligned}(\log\hat{\eta})_{-k} &= \frac{1}{2i\pi}\int \log\Big(z^{1/2}(z-e^{t})^{-1/2}(z-e^{-t})^{1/2}e^{i\pi/2}e^{t/2}\Big)z^{k}\frac{dz}{z}\\ &= \frac{1}{4i\pi}\int(\log z)z^{k}\frac{d}{z}+\frac{1}{4i\pi}\int\log(z-e^{t})^{-1/2}z^{k}\frac{dz}{z}+\frac{1}{4i\pi}\int\log(z-e^{-t})z^{k}\frac{dz}{z}\\ &+\frac{1}{4i\pi}\int\log e^{i\pi/2}z^{k}\frac{dz}{z}+\frac{1}{4i\pi}\int\log e^{t/2}z^{k}\frac{dz}{z}.\end{aligned} \tag{4.2.37}$$

Subsequently, we derive the integration for each term:

$$\begin{aligned}\frac{-1}{4i\pi}\int\log(z-e^{t})z^{k}\frac{dz}{z} &= \frac{-1}{4i\pi}\int\log\Big(z(1-e^{t}z^{-1})\Big)z^{k}\frac{dz}{z}\\ &= \frac{-1}{4i\pi}\int(\log z)z^{k}\frac{dz}{z}-\frac{1}{4i\pi}\int\log\Big(1-e^{t}z^{-1})z^{k}\frac{dz}{z}\\ &= \frac{-1}{4i\pi}\int(\log z)z^{k}\frac{dz}{z}+\frac{1}{4i\pi}\int\sum_{j=1}^{\infty}\frac{e^{tj}z^{-j}}{j}z^{k}\frac{dz}{z}\\ &= \frac{-1}{4i\pi}\int(\log z)z^{k}\frac{dz}{z}+\frac{1}{2k}e^{tk},\end{aligned} \tag{4.2.38}$$

and

$$\begin{aligned}\frac{1}{4i\pi}\int\log(z-e^{-t})z^{k}\frac{dz}{z} &= \frac{1}{4i\pi}\int\log\Big(-e^{-t}(1-e^{t}z)\Big)z^{k}\frac{dz}{z}\\ &= \frac{1}{4i\pi}\int\log e^{i\pi}z^{k}\frac{dz}{z}-\frac{1}{4i\pi}\int\log e^{t}z^{k}\frac{dz}{z}+\frac{1}{4i\pi}\int\log\Big(1-e^{t}z\Big)z^{k}\frac{dz}{z}\\ &= \frac{1}{4i\pi}\int\log e^{i\pi}z^{k}\frac{dz}{z}-\frac{1}{4i\pi}\int\log e^{t}z^{k}\frac{dz}{z}+\frac{1}{4i\pi}\int\sum_{j=1}^{\infty}\frac{e^{tj}z^{j}}{j}z^{k}\frac{dz}{z}=0.\end{aligned} \tag{4.2.39}$$

above eq, overfull! Hence, combining (4.2.38) and (4.2.39), we get:

$$(\log\hat{\eta})_{-k}=\frac{1}{2k}e^{tk}. \tag{4.2.40}$$

This leads us to

$$\begin{aligned} E(\hat{\eta}) &= \exp\left(\sum_{k\geq 1} k(\log\hat{\eta})_k(\log\hat{\eta})_{-k}\right) \\ &= \exp\left(\sum_{k\geq 1} k(\frac{1}{2k}e^{tk}\times\frac{-1}{2k}e^{tk})\right) \\ &= (1-e^{2t})^{1/4}. \end{aligned} \tag{4.2.41}$$

Consequently, the Toeplitz determinant for $\hat{\eta}(z)$ is given by:

$$\begin{aligned} D_n(\hat{\eta}(z)) &= G(\hat{\eta})^n E(\hat{\eta}) \\ &= e^{in\pi}\big(1-k^2\big)^{1/4}(1+o(1)). \end{aligned} \tag{4.2.42}$$

Then we will find $\frac{\hat{\phi}(0)}{\chi_n}$. By using the solution to the Riemann-Hilbert problem (3.3.1), we have:

$$\chi_{n-1}^2 = -Y_{21}(0),$$

and by using the steepest descent method as presented in the previous section, we have:

$$\begin{aligned} \chi_{n-1}^2 &= -D(0)^{-1}\left(1+\mathcal{O}\left(\frac{\rho^{-2n}}{1+|z|}\right)\right) \\ &= -e^{-i\pi}\left(1+\mathcal{O}\left(\frac{\rho^{-2n}}{1+|z|}\right)\right) \\ &= 1+\mathcal{O}\left(\frac{\rho^{-2n}}{1+|z|}\right), \end{aligned} \tag{4.2.43}$$

where $D(0) = G(\hat{\eta}) = e^{i\pi}$. Also, by the solution to the Riemann-Hilbert problem (3.3.1), we obtain:

$$Y_{21}(z) = -\chi_{n-1}z^{n-1}\hat{\phi}_{n-1}(z), \tag{4.2.44}$$

which gives us:

$$\frac{\hat{\phi}_{n-1}(0)}{\chi_{n-1}} = \frac{\lim_{z\to\infty}(Y_{21}(z;n)/z^{n-1})}{Y_{21}(0,n)}, \tag{4.2.45}$$

where

$$Y_{21}(0,n) = -\chi_{n-1}^2(I + o(1)). \tag{4.2.46}$$

Then, from (4.1.19) and (4.1.20), we obtain:

$$\begin{aligned} \frac{\hat{\phi}_{n-1}(0)}{\chi_{n-1}} &= -\frac{1}{\chi_{n-1}^2} \lim_{z\to\infty} \frac{D(z)z^n\left[R_{1,21}(z,n) + \mathcal{O}\left(\frac{\rho^{-3n}}{1+|z|}\right)\right]}{z^{n-1}} \\ &= -\frac{1}{\chi_{n-1}^2}\left(\lim_{z\to\infty} zR_{1,21}(z,n) + \mathcal{O}\left(\frac{\rho^{-3n}}{1+|z|}\right)\right), \end{aligned} \tag{4.2.47}$$

and by (4.1.18), we have:

$$\begin{aligned} -\lim_{z\to\infty} zR_{1,21}(z,n) &= -\lim_{z\to\infty} \frac{z}{2i\pi}\int_{\Gamma_1} \frac{\tau^{-n}\hat{\eta}^{-1}(\tau)D^{-2}(\tau)d\tau}{\tau - z} \\ &= \frac{1}{2i\pi}\int_{\Gamma_1} \tau^{-n}\hat{\eta}^{-1}(\tau)D^{-2}(\tau)d\tau. \end{aligned} \tag{4.2.48}$$

Then, using (4.2.20) and (4.2.28), and putting $\lambda = \tau^{-1}$ and $d\tau = -\frac{d\lambda}{\lambda^2}$, we obtain the following:

$$\begin{aligned} \frac{1}{2i\pi}\int_{\Gamma_1} \tau^{-n}\hat{\eta}^{-1}(\tau)D^{-2}(\tau)d\tau &= \frac{1}{2i\pi}\int_{\Gamma_0} \lambda^{n-\frac{3}{2}}\eta^{-1}(\lambda^{-1})D^{-2}(\lambda^{-1})d\lambda \\ &= \frac{-1}{2\pi\sqrt{k}}\int_{\Gamma_0} \lambda^{n-3/2}(\lambda-k)^{-1/2}(\lambda-k^{-1})^{-1/2}d\lambda, \end{aligned} \tag{4.2.49}$$

where $(\lambda - k^{-1})^{-1/2}$ is a holomorphic function at $\lambda = k$. Thus, we can write it as Taylor series

$$f(\lambda) = (\lambda - k^{-1})^{-1/2} = \sum_{l=0}^{\infty} d_l(\lambda - k)^l, \tag{4.2.50}$$

where

$$d_0 = \frac{-i\sqrt{k}}{\sqrt{1-k^2}}. \tag{4.2.51}$$

Now, we rewrite the right-hand side of (4.2.49) as follows:

$$\frac{-1}{2\pi\sqrt{k}}\int_{\Gamma_0}\lambda^{n-3/2}(\lambda-k)^{-1/2}(\lambda-k^{-1})^{-1/2}d\lambda = \frac{-1}{2\pi\sqrt{k}}\sum_{l=0}^{\infty}d_l\int_{\Gamma_0}\lambda^{n-3/2}(\lambda-k)^{l-1/2}d\lambda. \tag{4.2.52}$$

After that, we will shrink the contour Γ_0 to $[0,k]$, and we obtain:

$$\begin{aligned}\int_{\Gamma_0}\lambda^{n-3/2}(\lambda-k)^{l-1/2}d\lambda &= \frac{2}{i}\int_0^k x^{n-2}(x-k)^l\left(x^{1/2}(x-k)^{-1/2}\right)_+dx\\ &= \frac{2}{i}(-1)^l\int_0^k x^{n-3/2}(k-x)^{l-1/2}dx\\ &= \frac{2}{i}(-1)^l\int_0^1 (ky)^{n-3/2}(k-ky)^{l-1/2}kdy\\ &= \frac{2}{i}(-1)^l k^{n-3/2+l-1/2+1}\int_0^1 (y)^{n-3/2}(1-y)^{l-1/2}dy\\ &= \frac{2}{i}(-1)^l k^{n+l-1}\int_0^1 (y)^{n-3/2}(1-y)^{l-1/2}dy\\ &= \frac{2}{i}(-1)^l k^{n+l-1}B(n-1/2,l+1/2),\end{aligned} \tag{4.2.53}$$

where B is the beta function, which has the following asymptotics for large x, when y is fixed:

$$B(x,y)\sim\Gamma(y)x^{-y}.$$

Substituting (4.2.51) in (4.2.53), as $n\to\infty$ we obtain:

$$\begin{aligned}\frac{-1}{2\pi\sqrt{k}}\sum_{l=0}^{\infty}d_l\int_{\Gamma_0}\lambda^{n-3/2}(\lambda-k)^{l-1/2}d\lambda &= \frac{-1}{2\pi\sqrt{k}}\frac{2}{i}k^{n-1}\frac{-i\sqrt{k}}{\sqrt{1-k^2}}\times\Gamma(1/2)(n-1/2)^{-1/2}\\ &= k^{n-1}(n\pi)^{-1/2}(1-k^2)^{-1/2}+o(1).\end{aligned} \tag{4.2.54}$$

Therefore, we get:

$$\frac{\hat{\phi}_{n-1}(0)}{\chi_{n-1}} = k^{n-1}(n\pi)^{-1/2}(1-k^2)^{-1/2}. \tag{4.2.55}$$

Finally, by using (4.2.42) and (4.2.55), we get the asymptotic behaviour of the Toeplitz determinant as the following:

$$D_n(f(z;t)) = k^n (n\pi)^{-1/2}(1-k^2)^{-1/4} + o(1), \tag{4.2.56}$$

where $f(z;t)$ is the Fisher-Hartwig symbol (4.2.15) at $z=1$, with $\alpha = 0$, $\beta = -1$, and

$$o(1) = (\frac{\pi}{n})^{3/2} \frac{k^{n-1}}{4}(1-k^2)^{-3/2}.$$

4.3 Open problem

1. Studying the double-scaling limit for Toeplitz determinants in the low-temperature regime $t > 0$ has considered different types of symbols, as it has addressed in [13], [34], and in our work in Chapter 3. However, the high-temperature regime has not been considered yet for diagonal correlation functions.

2. Asymptotic of bordered Toeplitz determinants and next-to-diagonal Ising model denoted by $D_n^B[f,\psi]$, and is defined as follows:

$$D_n^B[f,\psi] = \det \begin{pmatrix} f_0 & f_1 & f_2 & \cdots & f_{(n-2)} & \psi_{(n-1)} \\ f_{-1} & f_0 & f_1 & \cdots & f_{(n-3)} & \psi_{n-2} \\ \vdots & \vdots & \ddots & \vdots & & \\ f_{1-n} & f_{2-n} & f_{3-n} & \cdots & f_{-1} & \psi_0 \end{pmatrix}, \quad n > 1 \tag{4.3.1}$$

   where f_n and ψ_n denote the Fourier coefficients of f and ψ, respectively. In [4], the authors studied the asymptotics of bordered Toeplitz determinants in the low-temperature regime $t > 0$, that is, $T < T_c$, by using both the operator theoretic and the Riemann-Hilbert approaches. They considered functions $f(z) = z^{-1}\eta(z)$ that possess Fisher-Hartwig singularities and

$$\psi = \frac{C_v \eta(z)}{S_v(z-c)}, \quad \text{with} \quad c = \frac{-S_h}{S_v} \tag{4.3.2}$$

   It is of importance to study the asymptotics of bordered Toeplitz determinants also in the high-temperature regime. However, in that case, the symbol $f(z)$ has a Fisher Hartwig singularity at $z = 1$ with $\alpha = 0, \beta = -1$, and the study involves the treatment of a Riemann Hilbert problem in the degenerate case $\alpha \pm \beta \in \mathbb{Z}_-$.

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